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HADWIGER'S CONJECTURE FOR K_6-FREE GRAPHS

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In 1943, Hadwiger made the conjecture that every loopless graph not contractible to the complete graph on $t+1$ vertices is t-colourable. When $t \leq 3$ this is easy, and when $t=4$, Wagner's theorem of 1937 shows the conjecture to be equivalent to the four-colour conjecture (the 4CC). However, when $t \geq 5$ it has remained open. Here we show that when $t=5$ it is also equivalent to the 4CC. More precisely, we show (without assuming the 4CC) that every minimal counterexample to Hadwiger's conjecture when $t = 5$ is "apex", that is, it consists of a planar graph with one additional vertex. Consequently, the 4CC implies Hadwiger's conjecture when $t = 5$, because it implies that apex graphs are 5-colourable.

1. Introduction

The following conjecture was made by H. Hadwiger in 1943 [4].

(1.1) (Hadwiger's conjecture) *For every $t \geq 0$, every loopless graph with no K_{t+1}-minor is t-colourable.*

(All graphs in this paper are finite; K_n is the complete graph with n vertices; a graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges; an *H-minor* of G is a minor isomorphic to H; a *t-colouring* of G is a function ϕ from the vertex set $V(G)$ of G into $\{1,\dots,t\}$ so that $\phi(u) \neq \phi(v)$ for every edge with ends u,v; and G is *t-colourable* if it has a t-colouring.)

For $t=0,1,2$ (1.1) is obvious, and Hadwiger [4] and Dirac [3] proved (1.1) for $t = 3$, when it is also easy. For $t = 4$, however, (1.1) seems extremely difficult. It evidently implies the four-colour conjecture (that every loopless planar graph is 4-colourable — briefly, the 4CC) because no planar graph has a K_5-minor; and in 1937 Wagner [17] proved the equivalence of the two. The 4CC remained open until 1977, when Appel and Haken [1, 2] gave a proof.

Our main result is that the 4CC implies Hadwiger's conjecture for $t=5$. Since the converse implication is easy, we cannot do without the 4CC. However, we can reformulate the main result to avoid mention of the 4CC, in the following way ((1.2)

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below). A graph G is *simple* if it has no loops or parallel edges. Let us say G is a *Hadwiger graph* if

(i) G is simple and not 5-colourable

(ii) every loopless minor of G with fewer vertices than G is 5-colourable, and

(iii) G has no K_6-minor (or equivalently, in view of (ii), $G \neq K_6$).

Hadwiger's conjecture for $t = 5$ is therefore that *there is no Hadwiger graph.* Let us say a graph G is *apex* if $G \setminus v$ is planar for some vertex v. (We use $G \setminus X$ to denote the graph obtained from G by deleting X; here X can be a vertex or an edge, or a set of vertices or edges.) Without assuming the 4CC, we shall prove the following.

(1.2) *Every Hadwiger graph is apex.*

Since the 4CC obviously implies that every loopless apex graph is 5-colourable and hence is not a Hadwiger graph, (1.2) together with the 4CC imply (1.1) with $t = 5$.

This paper is therefore devoted to proving (1.2). The proof falls into five separate steps. (We assume Mader's result that every Hadwiger graph is 6-connected.)

Step 1: *A non-apex Hadwiger graph has minimum valency* ≥ 7 *except for at most two vertices of valency* 6.

To prove this we study the distribution of K_4-subgraphs in a non-apex Hadwiger graph G. It is easy to show that no edge of G is in four triangles, and so no two K_4-subgraphs meet in exactly two vertices. If there are three K_4-subgraphs meeting pairwise in at most one vertex, then either they have a common vertex (when we can prove that G is apex, a contradiction, in section 3) or not (when we can find a K_6-minor, a contradiction, using Mader's "H-Wege" theorem, in section 4). Thus there are not three such subgraphs. On the other hand, it is easy to show that no three K_4-subgraphs meet pairwise in 3 vertices: and it follows that G has ≤ 4 K_4-subgraphs. But every vertex of valency 6 belongs to ≥ 2 K_4-subgraphs, for otherwise a 5-colouring of a minor of G could be extended to a 5-colouring of G; and it easily follows (in section 5) that there are ≤ 2 such vertices.

Step 2: *A non-apex Hadwiger graph is 7-connected except for its* (≤ 2) *vertices of valency* 6.

For this, assume that (A,B) is a *separation* of a non-apex Hadwiger graph G, that is, $A, B \subseteq V(G)$, $A \cup B = V(G)$, and no vertex in $A - B$ is adjacent to a vertex in $B - A$. Moreover, assume that $|A \cap B| = 6$, and $|A - B|$, $|B - A| \geq 2$. We prove in section 6 that for any four vertices $v_1, \ldots, v_4 \in A \cap B$, the restriction of G to $(A - B) \cup \{v_1, \ldots, v_4\}$ can be contracted to a K_4 on $\{v_1, \ldots, v_4\}$; this uses the result of step 1, and also a characterization of when such a contraction to K_4 is possible, proved in section 2. Now we examine the six-vertex graph $G|A \cap B$. (If $X \subseteq V(G)$, $G \mid X$ denotes the graph $G \setminus (V(G) - X)$.) It is easy to show, contracting K_4's from left and right onto $A \cap B$ appropriately, that $G|A \cap B$ has no circuit of length 4. The remainder of step 2 breaks into cases, because we need to enumerate all the possibilities for $G|A \cap B$. Here is a simple one, when $G|A \cap B$ has no edges: then we contract A to a single vertex, find a 5-colouring, and deduce that $G|B$ has a

5-colouring in which all the vertices in $A \cap B$ have the same colour. But so does $G|A$, and we fit these two 5-colourings together to obtain a 5-colouring of G, a contradiction. All except one of the possibilities for $G|A \cap B$ can be disposed of by this and similar arguments (section 7). The remaining possibility for $G|A \cap B$ is that it is a 5-edge path. Disposing of this is much more difficult, and occupies sections 8, 9 and 10; roughly we show that in this case, if we choose such (A, B) with A minimal, then both $G|A$ and $G|B$ can be drawn in the plane with ≤ 1 crossing, contrary to the result of step 1. This completes step 2.

Step 3: *Find ten forbidden subgraphs.*

We observed earlier that no edge was in four triangles. For this we only needed 6-connectivity, and now we have 7-connectivity (more or less) by step 2. We can therefore get more; for instance, that if we contract one edge of G, still no edge is in four triangles. By similar means, we find (in section 11) a list of ten graphs, with about 8 vertices and 11 edges, that are not subgraphs of any non-apex Hadwiger graph.

Step 4: *There is a perfect matching.*

More exactly, there is a matching of cardinality $\left\lfloor \frac{1}{2} V(G) \right\rfloor$. For if not, by Tutte's theorem, there exists $Z \subseteq V(G)$ such that $G \setminus Z$ has $\geq |Z| + 2$ components, and by contracting appropriately we obtain a simple minor H of G with $\geq 4|V(H)|$ edges; but this is impossible, for Mader proved that a simple graph H with $> 4|V(H)| - 10$ edges has a K_6-minor. This is the content of section 12.

Step 5: *There is a reducible configuration.*

By a "reducible configuration" we mean, roughly, a subgraph of G (whose vertices typically have small valency in G) such that there corresponds a proper minor of G every 5-colouring of which induces a 5-colouring of G. The most trivial one is a single vertex v which is 4-valent in G; then every 5-colouring of $G \backslash v$ extends to one of G. Of course, we already know that G has no 4-valent vertices, but there are more useful reducible configurations, for example, two adjacent vertices of valency 7 and 8, joined by an edge in three triangles, where neither vertex is in a K_4-subgraph. A Hadwiger graph by definition cannot contain a reducible configuration. However, let us take the matching of step 4, and contract its edges, and delete any resultant parallel edges. If $|V(G)| = n$, we obtain a graph with (about) $\frac{1}{2}n$ vertices, and therefore, by Mader's theorem, at most $4\left(\frac{1}{2}n\right) - 10$ edges. But G has $\geq \frac{7}{2}n - 1$ edges, by step 1; where did the extra $> \frac{3}{2}n$ edges go? $\frac{1}{2}n$ were lost because they were contracted, but the remaining $> n$ edges became parallel and were discarded for that reason. Consequently, on average an edge of the matching belongs to several triangles or squares, and more (on average) if its ends have valency > 7. This leads to a proof (in section 13) that there is either a reducible configuration or a forbidden subgraph in any non-apex Hadwiger graph, and so there is no such graph.

There is a very interesting conjecture due to Jørgensen [5], that every 6-connected graph with no K_6-minor is apex. This would obviously imply our result, because Hadwiger graphs are 6-connected, and we spent a good deal of effort trying to prove it, with no success. However, it does seem to us to be true, and with a view to this conjecture we organized sections 2-4 to apply to all graphs satisfying the hypotheses of the conjecture, rather than just to Hadwiger graphs.

2. Finding a K_4-minor

Let G be a graph. Its vertex- and edge-sets are denoted by $V(G)$ and $E(G)$. As in section 1, $G \setminus X$ denotes the result of deleting X, and for $X \subseteq V(G), G \mid X$ denotes $G \setminus (V(G) - X)$. Thus, $G|X$ is the subgraph of G induced on X. A subset $X \subseteq V(G)$ is a *fragment* of G if $X \neq \emptyset$ and $G|X$ is connected. If $X, Y \subseteq V(G)$, we say XY are *adjacent* in G if $X \cap Y = \emptyset$ and some $x \in X$ is adjacent in G to some $y \in Y$. If there is an edge of G with ends $x, y \in V(G)$ we say xy are adjacent (with no comma, because we shall need lists $ab, uv, xy, \ldots$ of adjacent pairs), and if there is a unique edge with ends x, y we speak of the edge xy or yx.

A *cluster* in G is a set of mutually adjacent fragments of G, and it is a *p-cluster* if it has cardinality p. Thus, G has a K_p-minor if and only if it has a p-cluster. Given p distinct vertices $v_1, \ldots, v_p$ a cluster $\mathscr{C}$ is said to *traverse* $v_1, \ldots, v_p$ or $\{v_1, \ldots, v_p\}$ if $|\mathscr{C}| = p$, and $\mathscr{C}$ can be written as $\mathscr{C} = \{X_1, \ldots, X_p\}$ in such a way that $v_i \in X_i$ $(1 \leq i \leq p)$. Our concern here is, given four vertices of a graph G, when is there a cluster in G traversing them?

If H, J are subgraphs of G, then $H \cup J$ denotes the subgraph with vertex set $V(H) \cup V(J)$ and edge set $E(H) \cup E(J)$, and $H \cap J$ is defined similarly. We say subgraphs H, J are *disjoint* if $V(H \cap J) = \emptyset$. A *separation* of G is a pair (A, B) of subsets of $V(G)$ such that $(G \mid A) \cup (G \mid B) = G$, that is, $A \cup B = V(G)$ and no edge has one end in $A - B$ and the other in $B - A$. Its *order* is $|A \cap B|$. It is a *k-separation* if it has order k, and a *$(\leq k)$-separation* if its order is $\leq k$.

Let $Z_1, Z_2, \ldots, Z_k \subseteq V(G)$ be disjoint. We say that (the subpartition) $Z_1, \ldots, Z_k$ is *feasible* in G (*via* $X_1, \ldots, X_k$) if there are disjoint fragments $X_1, \ldots, X_k$ of G with $Z_i \subseteq X_i$ $(1 \leq i \leq k)$; and it is *infeasible* otherwise.

Paths and *circuits* by definition have no repeated vertices or edges. We begin with the following.

(2.1) *Let $v_1, \ldots, v_4 \in V(G)$ be distinct. Then there exist disjoint fragments $X_1, \ldots, X_4$ of G such that $v_i \in X_i$ $(1 \leq i \leq 4)$ and $X_1X_2, X_2X_3, X_3X_4, X_4X_1$ are adjacent, if and only if $\{v_1, v_2\}$, $\{v_3, v_4\}$ and $\{v_2, v_3\}, \{v_1, v_4\}$ are both feasible in G.*

Proof. The "only if" implication is easy, and we prove "if". Let P, Q, R, S be paths of G, chosen with $P \cup Q \cup R \cup S$ minimal, such that

(i) P has ends v_1v_2, Q has ends v_2v_3, R has ends v_3v_4, and S has ends v_1v_4

(ii) P, R are disjoint and Q, S are disjoint.

These exist, from the feasibility hypothesis.

By an *arc* we mean here a path of $Q \cup S$ with distinct ends both in $P \cup R$ and with no edge or internal vertex in $P \cup R$. Every arc is a subpath of Q or of S, and both Q and S contain at least one arc.

(1) *Every arc has one end in $V(P)$ and the other in $V(R)$.*

For if some arc A has both ends in $V(P)$ say, let P' be the path obtained from P by replacing by A the subpath of P between the ends of A; then P',Q,R,S satisfy (i) and (ii) above, and $P' \cup Q \cup R \cup S$ is a proper subgraph of $P \cup Q \cup R \cup S$, contrary to the choice of P,Q,R,S.

Let Q' be the arc in Q closest to v_2, with ends a,b where a lies in Q between b and v_2. Let P' be the subpath of Q between a and v_2. Since $P' \subseteq P \cup R$ and has an end v_2, it follows that $P' \subseteq P \cap Q$, and in particular $a \in V(P)$ and $b \in V(R)$ by (1). Let S' be the arc in S closest to v_4, with ends c,d where c lies between d and v_4; and let R' be the subpath of S between c and v_4. Similarly, $R' \subseteq R \cap S, c \in V(R)$, and $d \in V(P)$. Now $d \notin V(P')$ since $P' \subseteq Q$ and $d \notin V(Q)$; and $b \notin V(R')$ similarly. Thus taking

$$\begin{aligned} X_1 &= V(P) - V(P') \\ X_2 &= V(P' \cup Q') - \{b\} \\ X_3 &= V(R) - V(R') \\ X_4 &= V(R' \cup S') - \{d\} \end{aligned}$$

satisfies the theorem. ∎

In (2.1) we asked that a specific four pairs of $X_1,\ldots,X_4$ should be adjacent. Eventually, we want all six to be adjacent; and the next step is a specific five. A *trisection* of G is a triple (A,B,C) of subsets of $V(G)$ such that $A \cap B = A \cap C = B \cap C$ and $(G|A) \cup (G|B) \cup (G|C) = G$; its *order* is $|A \cap B \cap C|$.

(2.2) *Let $v_1,\ldots,v_4 \in V(G)$ be distinct. Then the following are equivalent:*

- (i) *there exist disjoint fragments $X_1,\ldots,X_4$ of G with $v_i \in X_i$ $(1 \le i \le 4)$ such that $X_1X_2, X_1X_3, X_1X_4, X_2X_3, X_2X_4$ are adjacent*
- (ii) *all the following hold:*
  - (a) *$\{v_1,v_3\},\{v_2,v_4\}$ is feasible,*
  - (b) *$\{v_1,v_4\},\{v_2,v_3\}$ is feasible, and*
  - (c) *for every trisection (A_1,A_2,B) of G of order 2 with $A_1 \cap A_2 \cap B = \{x_1,x_2\}$ such that $v_i \in A_i - \{x_1,x_2\}$ $(i = 1,2)$ and $v_3,v_4 \in B$, there are disjoint fragments $Y_1,\ldots,Y_4$ of $G \mid B$ with $x_1 \in Y_1, x_2 \in Y_2$, $v_3 \in Y_3, v_4 \in Y_4$ such that Y_1Y_3, Y_1Y_4, Y_2Y_3, Y_2Y_4 are all adjacent.*

Proof. That (i) implies (ii) is easy, and we omit it. Let us prove the converse. We assume that (ii) holds. By (2.1) and (ii)(a), (ii)(b), there is a circuit C of G, and four distinct vertices u_1,u_2,u_3,u_4 of C, such that u_1,u_3,u_2,u_4 occur in C in order, and there are four disjoint paths $P_1,\ldots,P_4$ of G, such that P_i has ends u_i,v_i and has no vertex in C except u_i. Choose C and $P_1,\ldots,P_4$ with $P_3 \cup P_4$ minimal. Let the path of C between u_1 and u_3 not containing u_2,u_4 be C_{13}, and define C_{14},C_{23},C_{24} similarly.

(1) *There is no path of G from $V(P_1 \cup P_2 \cup C)$ to $V(P_3 \cup P_4)$ with no vertex in $\{u_3,u_4\}$.*

For if there is such a path P we may assume it has one end u in $V(P_1 \cup C_{13} \cup C_{14})$ and the other end v in $V(P_3)$, and has no vertex in $\{u_3,u_4\}$, and has no vertex in

$C \cup P_1 \cup P_2 \cup P_3 \cup P_4$ except its ends. If $u \in V(P_1 \cup C_{14})$ we may replace C_{13} by P, contrary to the minimality of $P_3 \cup P_4$; and if $u \in V(C_{13})$ we replace the subpath of C_{13} between u and u_3 by P, again contrary to the minimality of $P_3 \cup P_4$. This proves (1).

From (1), there is a separation (A,B) of G with $V(C \cup P_1 \cup P_2) \subseteq A$ and $V(P_3 \cup P_4) \subseteq B$, with $A \cap B = \{u_3, u_4\}$.

(2) *We may assume that there is a separation* (A_1, A_2) *of* $G|A$ *with* $A_1 \cap A_2 = \{u_3, u_4\}$, $v_1 \in A_1 - \{u_3, u_4\}$ *and* $v_2 \in A_2 - \{u_3, u_4\}$.

For if there is a path of $G|A$ from v_1 to v_2 avoiding u_3 and u_4, there is a minimal path P from $V(P_1 \cup C_{13} \cup C_{14}) - \{u_3, u_4\}$ to $V(P_2 \cup C_{23} \cup C_{24}) - \{u_3, u_4\}$ in $G|(A - \{u_3, u_4\})$; but then taking

$$\begin{aligned} X_1 &= V(P_1 \cup C_{13} \cup C_{14} \cup P) - \{u_3, u_4, v\} \\ X_2 &= V(P_2 \cup C_{23} \cup C_{24}) - \{u_3, u_4\} \\ X_3 &= V(P_3) \\ X_4 &= V(P_4) \end{aligned}$$

satisfies (i) where v is the end of P in $V(P_2 \cup C_{23} \cup C_{24}) - \{u_3, u_4\}$. This proves (2).

From (ii)(c) applied to the trisection (A_1, A_2, B) of (2), there are disjoint fragments $Y_1, \ldots, Y_4$ of $G|B$ such that $u_3 \in Y_1$, $u_4 \in Y_2$, $v_3 \in Y_3$, $v_4 \in Y_4$, and Y_1Y_3, Y_1Y_4, Y_2Y_3, Y_2Y_4 are all adjacent. Let

$$\begin{aligned} X_1 &= Y_1 \cup V(P_1 \cup C_{13} \cup C_{14}) - \{u_4\} \\ X_2 &= Y_2 \cup V(P_2 \cup C_{24}) \\ X_3 &= Y_3 \\ X_4 &= Y_4; \end{aligned}$$

then (i) holds, as required. ∎

The main result of this section is the following.

(2.3) *Let* $Z \subseteq V(G)$ *with* $|Z| = 4$. *Then the following are equivalent:*

(i) *there is a cluster in* G *traversing* Z

(ii) *for every ordering* $Z = \{v_1, \ldots, v_4\}$ *both the following hold:*

(a) $\{v_1, v_2\}, \{v_3, v_4\}$ *is feasible in* G, *and*

(b) *for every trisection* (A_1, A_2, B) *of* G *of order 2 with* $A_1 \cap A_2 \cap B = \{x_1, x_2\}$ *such that* $v_i \in A_i - \{x_1, x_2\}$ $(i = 1, 2)$ *and* $v_3, v_4 \in B$, *there are disjoint fragments* $Y_1, \ldots, Y_4$ *of* $G \mid B$ *with* $x_1 \in Y_1, x_2 \in Y_2$, $v_3 \in Y_3, v_4 \in Y_4$ *such that* Y_1Y_3, Y_1Y_4, Y_2Y_3, Y_2Y_4, Y_3Y_4 *are all adjacent.*

Proof. Again, that (i) implies (ii) is easy, and we shall just prove the converse. We assume that (ii) holds. It follows easily that we may assume G is 2-connected (by induction on $|V(G)|$, say). We assume for a contradiction that (i) is false.

(1) *There is no trisection (A_1,A_2,B) of order 2 such that $A_1-(A_2\cup B)$ and $A_2-(A_1\cup B)$ both contain exactly one member of Z.*

For suppose that (A_1,A_2,B) is such a trisection, and let $A_1\cap A_2\cap B=\{x_1,x_2\}$. Let $Z=\{v_1,\dots,v_4\}$ where $v_i\in A_i-\{x_1,x_2\}$ $(i=1,2)$ say. Since G is 2-connected, $\{v_1,x_1\}$ is feasible in $G|(A_1-\{x_2\})$, and $\{v_2,x_2\}$ is feasible in $G|(A_2-\{x_1\})$, and hence $\{v_1,x_1\},\{v_2,x_2\}$ is feasible in $G|(A_1\cup A_2)$. Also, since G is 2-connected, there is a path of $G|(A_1\cup A_2)$ between v_1 and v_2. Consequently, there are adjacent fragments Y_1',Y_2' of $G|(A_1\cup A_2)$ with $v_1,x_1\in Y_1'$ and $v_2,x_2\in Y_2'$. By (ii) there are fragments $Y_1,\dots,Y_4$ of $G|B$ as in (ii). Then $\{Y_1\cup Y_1',Y_2\cup Y_2',Y_3,Y_4\}$ is a cluster in G traversing Z, a contradiction. This proves (1). ∎

(2) *There is no 2-separation (A,B) of G such that $|(A-B)\cap Z|=|(B-A)\cap Z|=2$.*

For suppose that (A,B) is such a 2-separation; let $(A-B)\cap Z=\{v_1,v_2\}$, $(B-A)\cap Z=\{v_3,v_4\}$, $A\cap B=\{x_1,x_2\}$. By exchanging v_1,v_2 if necessary, we may assume that $\{v_1,x_1\},\{v_2,x_2\}$ is feasible in $G|A$, since G is 2-connected, and similarly that $\{v_3,x_1\},\{v_4,x_2\}$ is feasible in $G\mid B$.

By (2.3)(ii)(a), $\{v_1,v_4\},\{v_2,v_3\}$ is feasible in G, and so either $\{v_1,x_2\},\{v_2,x_1\}$ is feasible in $G\mid A$, or $\{v_3,x_2\},\{v_4,x_1\}$ is feasible in $G\mid B$, and from the symmetry we may assume the latter. If (A_1',A_2',B') is a trisection of $G\mid B$ of order 2 with $A_1'\cap A_2'\cap B'=\{x_1',x_2'\}$ say, and with $v_3\in A_1'-\{x_1',x_2'\}$, $v_4\in A_2'-\{x_1',x_2'\}$ and $x_1,x_2\in B'$, then $(A_1',A_2',B'\cup A)$ is a trisection of G contrary to (1). Thus there is no such (A_1',A_2',B'), and so by (2.2) applied to $G\mid B$, there are disjoint fragments $Y_1,\dots,Y_4$ of $G\mid B$ with $x_1\in Y_1,x_2\in Y_2$, $v_3\in Y_3,v_4\in Y_4$, and with $Y_1Y_3,Y_1Y_4,Y_2Y_3,Y_2Y_4,Y_3Y_4$ all adjacent. Choose disjoint fragments Y_1',Y_2' of $G\mid A$ with $v_1,x_1\in Y_1'$ and $v_2,x_2\in Y_2'$ and with $Y_1'Y_2'$ adjacent (this is possible since G is 2-connected); then $\{Y_1\cup Y_1',Y_2\cup Y_2',Y_3,Y_4\}$ satisfies (i), a contradiction. This proves (2).

(3) *There do not exist disjoint paths P_1,P_2 of G with ends $v_1,v_3\in Z$ and $v_2,v_4\in Z$ respectively, and distinct vertices a_1,b_1,c_1 of P_1 in order (with a_1 closest to v_1) and distinct vertices a_2,b_2,c_2 of P_2 in order (with a_2 closest to v_2) and disjoint paths Q_1,Q_2,Q_3 of G with ends a_1b_2,a_2b_1, and c_1c_2 respectively, so that Q_1,Q_2,Q_3 have no vertices in $V(P_1\cup P_2)$ except their ends.*

For suppose such P_1,P_2,Q_1,Q_2,Q_3 exist. Since $b_1,b_2\neq v_1,v_2,v_3,v_4$, there is by (2) a path P of G from

$$V(A_1\cup Q_1\cup B_1\cup A_2\cup Q_2\cup B_2)$$

to $V(C_1\cup D_1\cup C_2\cup D_2\cup Q_3)$, with $b_1,b_2\notin V(P)$, where A_1,B_1,C_1,D_1 are the subpaths of P_1 with ends $v_1a_1,a_1b_1,b_1c_1,c_1v_3$ and $A_2,B_2,C_2,D_2\subseteq P_2$ are defined similarly. Take a minimal such subpath P, with ends $u\in V(A_1\cup B_1\cup Q_1)-\{b_1,b_2\}$ and $v\in V(C_1\cup D_1\cup Q_3)-\{b_1,c_2\}$ say (without loss of generality, by exchanging v_1 with v_2 or v_3 with v_4). Let

$$X_1=V(A_1\cup B_1\cup Q_1)-\{b_1,b_2\}$$
$$X_2=V(A_2\cup B_2\cup Q_2)-\{b_2\}$$
$$X_3=V(C_1\cup D_1\cup Q_3\cup P)-\{b_1,c_2,u\}$$
$$X_4=V(C_2\cup D_2);$$

then $\{X_1,\ldots,X_4\}$ is a cluster traversing Z, a contradiction. This proves (3).

Let $Z=\{v_1,\ldots,v_4\}$. Since $\{v_1,v_2\},\{v_3,v_4\}$ is feasible and so are the other two similar partitions, it follows from (2.1) that there is a circuit C and four distinct vertices u_1,u_2,u_3,u_4 of it, in order on C, and four disjoint paths $P_1,\ldots,P_4$, where P_i has ends v_i,u_i and has no vertex in C except u_i; and there are disjoint paths Q,R with ends v_1,v_3 and v_2,v_4 respectively. Let $P_1\cup\ldots\cup P_4\cup C=H$. Let C_{12} be the path of C between u_1 and u_2 not containing u_3,u_4, and define C_{23},C_{34},C_{41} similarly. By an *arc* we mean a subpath of $Q\cup R$ with distinct ends both in $V(H)$ and with no edge or internal vertex in H.

(4) *No arc has ends $u\in V(P_1)-\{u_1\}$ and $v\in V(C_{23}\cup C_{34}\cup P_3)-\{u_2,u_4\}$.*

For suppose that P is such an arc. By (3) (with v_1,v_4 exchanged) $v\notin V(P_3)-\{u_3\}$; by (3) $v\notin V(C_{23})-\{u_2,u_3\}$, and by (3) (with v_2,v_4 exchanged) $v\notin V(C_{34})-\{u_3,u_4\}$. Thus, $v=u_3$. Let $T_1=P\cup P_1,T_2=C_{12}\cup C_{23}\cup P_2$, $T_3=C_{41}\cup C_{34}\cup P_4$; we see there is symmetry between T_1,T_2 and T_3 exchanging v_1,v_2 and v_4 and fixing u_1. By (1) there is a path S of G joining two of T_1,T_2,T_3,P_3 with no vertex in $\{u_1,u_3\}$. Choose a minimal such path S, with ends a,b say. From (3) with $v_1,\ldots,v_4$ permuted, it follows that $a,b\notin V(P_3)$, and so we may assume from the symmetry that $a\in V(T_1)$ and $b\in V(T_2)$. Then setting

$$\begin{aligned}
X_1 &= V(S\cup T_1)-\{u_1,u_3,b\}\\
X_2 &= V(T_2)-\{u_1,u_3\}\\
X_3 &= V(P_3)\\
X_4 &= V(T_3)-\{u_3\}
\end{aligned}$$

defines a cluster traversing Z, a contradiction. This proves (4).

Now choose $P_1,\ldots,P_4,C,Q,R,H$ with $H\cup Q\cup R$ minimal, and subject to that with $\sum|E(P_i)|$ minimum.

(5) *No arc has an end in $V(P_1)-\{u_1\}$.*

For suppose that P is an arc with ends u,v where $u\in V(P_1)-\{u_1\}$. By (4),

$$v\in V(P_1\cup C_{12}\cup P_2\cup C_{41}\cup P_4)-\{u\},$$

and by the symmetry we may assume that $v\in V(P_1\cup C_{12}\cup P_2)-\{u\}$. If $v\in V(P_1)$, then we may replace the subpath of P_1 between u and v by P, thereby reducing the union $H\cup Q\cup R$, a contradiction. If $v\in V(C_{12}\cup P_2)-\{u_1\}$, we may replace by P either C_{12} (if $v\in V(P_2)$) or the path of C_{12} between u_1 and v (if $v\in V(C_{12})$), thereby reducing $\sum|E(P_i)|$ while not increasing the union $H\cup Q\cup R$, a contradiction. This proves (5).

From (5) it follows that $P_1\subseteq Q$, and similarly $P_2\subseteq R,P_3\subseteq Q,P_4\subseteq R$. Now $Q\not\subseteq H$ since $u_2,u_4\notin V(Q)$ and so there is an arc in Q; let the first arc in Q be A (that is, closest to v_1 in Q). Similarly, let the arc in R closest to v_2 be B. Let A have ends a_1,a_2, and B have ends b_1,b_2, where a_1 is between v_1 and a_2 in Q, and b_1 is between v_2 and b_2 in R. Since the subpath Q' of Q between v_1 and a_1 is in H, it follows that $a_1\in V(C_{12}\cup C_{14})-\{u_2,u_4\}$, and Q' is the path of $H\setminus\{u_2,u_4\}$

between v_1 and a_1. Suppose that $a_2 \in V(C_{12} \cup C_{41})$. Let $a_2 \in V(C_{12}) - \{u_1, u_2\}$ say. If $a_1 \in V(C_{12})$ we may reduce the union $H \cup Q \cup R$ by replacing the subpath of C_{12} between a_1 and a_2 by A; and if $a_1 \in V(C_{41})$, we may similarly reduce the union by replacing by A either the subpath of C_{12} between u_1 and a_2, or the subpath of C_{41} between u_1 and a_1, whichever is not included in $Q \cup R$. In either case, we have a contradiction, and so $a_2 \notin V(C_{12} \cup C_{41})$. Hence, $a_2 \in V(C_{23} \cup C_{34})$. By exchanging v_2 and v_4 we may therefore assume that $a_2 \in V(C_{23})$. Similarly, $b_1 \in V(C_{12} \cup C_{23})$ and $b_2 \in V(C_{34} \cup C_{41})$. Let R' be the subpath of R between v_2 and b_1. Then setting

$$X_1 = V(Q' \cup C_{12}) - V(R')$$
$$X_2 = V(R')$$
$$X_3 = V(C_{23} \cup P_3 \cup A) - (V(R') \cup \{a_1\})$$
$$X_4 = V(C_{34} \cup C_{41} \cup P_4 \cup B) - (V(Q') \cup \{b_1, u_3\})$$

defines a cluster traversing Z, a contradiction. This completes the proof. ∎

We need also the following, a slight variation on a result of [12] — see also [6, 13, 14, 15].

(2.4) *Let $v_1, \ldots, v_k$ be distinct vertices of a graph G. Then either*

(i) *there are disjoint paths of G with ends p_1p_2 and q_1q_2 respectively, so that p_1, q_1, p_2, q_2 occur in the sequence $v_1, \ldots, v_k$ in order, or*

(ii) *there is a (≤ 3)-separation (A, B) of G with $v_1, \ldots, v_k \in A$ and $|B - A| \geq 2$, or*

(iii) *G can be drawn in a disc with $v_1, \ldots, v_k$ on the boundary in order.*

Proof. We may assume that every vertex of G not in $\{v_1, \ldots, v_k\}$ has ≥ 3 neighbours. Hence there is no (≤ 2)-separation (A, B) of G with $v_1, \ldots, v_k \in A$ and $|B - A| = 1$, and the statement follows from [12, theorems (2.3) and (2.4)]. ∎

We deduce

(2.5) *Let $s_1, t_1, s_2, t_2 \in V(G)$ be distinct. Then either*

(i) *$\{s_1, t_1\}, \{s_2, t_2\}$ is feasible in G, or*

(ii) *$\{s_1, t_1\}$ is not feasible in $G \backslash \{s_2, t_2\}$, or $\{s_2, t_2\}$ is not feasible in $G \backslash \{s_1, t_1\}$, or*

(iii) *there is a (≤ 3)-separation (A, B) of G with $s_1, t_1, s_2, t_2 \in A$ and $|B - A| \geq 2$ and $|B \cap \{s_1, t_1, s_2, t_2\}| \leq 2$, or*

(iv) *G can be drawn in a disc with s_1, s_2, t_1, t_2 on the boundary in order.*

Proof. We proceed by induction on $|V(G)| + |E(G)|$. We may therefore assume that G is simple, and every vertex not in $\{s_1, s_2, t_1, t_2\}$ has valency ≥ 3. By (2.4), since we may assume that (i) and (iv) are false, there is a (≤ 3)-separation (A, B) of G with $s_1, t_1, s_2, t_2 \in A$ and $|B - A| \geq 2$. We may therefore assume that B contains three of s_1, t_1, s_2, t_2, for otherwise (iii) holds; say $s_1, t_1, s_2 \in B$. Assuming (ii) is false, there is a path P from s_2 to t_2 with $s_1, t_1 \notin V(P)$ and hence with $V(P) \cap B = \{s_2\}$. Since (i) is false, there is no path in $G \mid (B - \{s_2\})$ between s_1

and t_1. Consequently we may choose a separation (X,Y) of $G \mid B$ with $X \cap Y = \{s_2\}, s_1 \in X$ and $t_1 \in Y$. Since $|B - A| \geq 2$, we may assume that $|X - A| \geq 1$; let $v \in X - A$. Since v has valency ≥ 3, it follows that $|X| \geq 4$, and so $|X - A| \geq 2$. But $(Y \cup A, X)$ is a 2-separation of G with $s_1, t_1, s_2, t_2 \in Y \cup A$, and so (iii) holds. ∎

From (2.3) and (2.5) we deduce:

(2.6) *Let $Z \subseteq V(G)$ with $|Z| = 4$. Then either*

(i) *there is a cluster in G traversing Z, or*

(ii) *there is a trisection (A_1, A_2, B) of order 2 such that $|Z \cap (A_i - B)| = 1$ $(i = 1,2)$, or*

(iii) *there is a (≤ 3)-separation (A,B) with $Z \subseteq A$ and $|B - A| \geq 2$ and $|Z \cap B| \leq 2$, or*

(iv) *G can be drawn in a plane so that every vertex in Z is incident with the infinite region.*

Proof. We assume that (i) is false. By (2.3) we may order $Z = \{v_1, \ldots, v_4\}$ so that one of (2.3)(ii)(a), (2.3)(ii)(b) is false. If (2.3)(ii)(a) is false, then by (2.5), one of (ii), (iii), (iv) holds. (In particular, if (2.5)(ii) holds and $Z = \{v_1, \ldots, v_4\}$ and $\{v_1, v_2\}$ is not feasible in $G \setminus \{v_3, v_4\}$, then (ii) holds, taking $B = \{v_3, v_4\}$.) If (2.3)(ii)(b) is false, then (ii) holds. ∎

We shall apply (2.6) several times in our approach to Hadwiger's conjecture; the first is the following, which for Hadwiger graphs was proved independently by J. Mayer (unpublished). A *triangle* of G is a circuit of G of length 3.

(2.7) *Let G be a simple 6-connected graph with no K_6-minor, which is not apex. Then every edge of G is in ≤ 3 triangles.*

Proof. Suppose that there are triangles with vertex sets $\{x_1, x_2, v_i\}$ $(1 \leq i \leq 4)$, where $x_1, x_2, v_1, v_2, v_3, v_4$ are distinct. Let $G' = G \setminus \{x_1, x_2\}$, and let us apply (2.6) to G', taking $Z = \{v_1, \ldots, v_4\}$. If (2.6)(i) holds, and $\mathscr{C}$ is a cluster in G' traversing Z, then $\mathscr{C} \cup \{\{x_1\}, \{x_2\}\}$ is a 6-cluster in G, a contradiction. Since G' is 4-connected, (2.6)(ii) and (2.6)(iii) do not hold, and so (2.6)(iv) holds, and G' can be drawn in a plane so that $v_1, \ldots, v_4$ are all incident with the infinite region. Since G' is 2-connected and loopless and $|V(G')| \geq 3$ there is a circuit C bounding the infinite region. Let $X = V(G') - V(C)$. Since G' is 3-connected and $|V(C)| \geq 4$, it follows that $X \neq \emptyset$, and that X is a fragment of G'. Let P_1, P_2, P_3 be three disjoint paths in C with $V(P_1 \cup P_2 \cup P_3) = V(C)$ and with $v_i \in V(P_i)$ $(1 \leq i \leq 3)$. Now v_1, v_2, v_3 all have neighbours in X since G' is 3-connected, and yet

$$\{V(P_1), V(P_2), V(P_3), X, \{x_1\}, \{x_2\}\}$$

is not a 6-cluster in G. Consequently, one of x_1, x_2 has no neighbour in X, say x_2. But then $G \setminus x_1$ is planar, and so G is apex, a contradiction as required. ∎

Let us also mention the following, the proof of which is clear.

(2.8) *Let G be a 5-connected graph with no K_6-minor and with $|V(G)| \geq 6$. Then no subgraph of G is isomorphic to K_5.*

3. Triads and tripods

A *triad* in G is a connected subgraph T of G with no circuits, with one vertex of valency 3 and all others of valency ≤ 2. Necessarily, it has precisely three vertices of valency 1, called its *feet*. It is *lean* (in G) if $V(T') = V(T)$ for every triad T' in G with $V(T') \subseteq V(T)$ and with the same feet as T.

If H is a subgraph of G, an *H-flap* is the vertex set of a connected component of $G \setminus V(H)$.

(3.1) *Let G be simple and let $v_1, v_2, v_3 \in V(G)$ be distinct, such that there is no (≤ 3)-separation (A, B) with $v_1, v_2, v_3 \in A$, $|A| \geq 4$ and $|V(G) - A| \geq 1$. Let T_0 be a triad in G with feet v_1, v_2, v_3, and let W be a T_0-flap. Then there is a lean triad T with feet v_1, v_2, v_3 and with $V(T) \cap W = \emptyset$, such that there is only one T-flap.*

Proof. (Our thanks to the referee for the following, which is much better than our original proof.) We say $(\alpha_1, \ldots, \alpha_n)$ is *lexicographically larger* than $(\beta_1, \ldots, \beta_m)$ if either

(i) $m < n$ and $\alpha_i = \beta_i$ for $i = 1, \ldots, m$, or

(ii) there exists j with $1 \leq j \leq min(m, n)$ so that $\alpha_j > \beta_j$ and $\alpha_i = \beta_i$ for $i = 1, \ldots, j-1$.

Let T be a triad with feet v_1, v_2, v_3, and with $V(T) \cap W = \emptyset$. Since $G \mid W$ is connected, there is a T-flap B_1 say, with $W \subseteq B_1$; let the T-flaps be $B_1, \ldots, B_n$ say, ordered so that $|B_2| \geq |B_3| \geq \ldots \geq |B_n|$. Since there is such a triad T (namely T_0), we may choose T so that $(|B_1|, \ldots, |B_n|)$ is lexicographically maximum. We shall show that T satisfies the theorem. Clearly it is lean; we must show that $n = 1$.

Let us say that $v \in V(G)$ is *essential* if $v \in V(T')$ for every triad T' with feet v_1, v_2, v_3 and with $V(T') \subseteq V(T) \cup B_n$. Every vertex v of $V(T) \cup B_n$ with a neighbour in $B_1 \cup \ldots \cup B_{n-1}$ is essential: for if not, there is a triad T' with feet v_1, v_2, v_3 and with $V(T') \subseteq (V(T) \cup B_n) - \{v\}$, and replacing T by T' would give a lexicographic increase of $(|B_1|, \ldots, |B_n|)$.

Let S be the set of all essential vertices; thus, $S \subseteq V(T)$. Let K be the component of $G \setminus S$ containing B_n; thus, $V(K) \subseteq V(T) \cup B_n$. Let S' be the set of all vertices in S with a neighbour in K.

We claim that $|S'| \leq 3$. If $S' \subseteq \{v_1, v_2, v_3\}$ the claim is true, and so we may assume that $S' \cap (V(T) - \{v_1, v_2, v_3\}) \neq \emptyset$. Consequently, for $i = 1, 2, 3$ there is a path of T from v_i to a member of S' with only one vertex in $\{v_1, v_2, v_3\}$; and by choosing it minimal we may assume it has only one vertex in S'. Consequently there is a path P_i from v_i to $V(K)$ with

$$V(P_i) \subseteq V(T) \cup V(K) \subseteq V(T) \cup B_n$$

with only one vertex in $\{v_1, v_2, v_3\}$. Now $P_1 \cup P_2 \cup P_3 \cup K$ is connected and v_1, v_2, v_3 each have valency 1 in this subgraph; for $v_i \notin V(K)$ since $v_i \in S$ $(1 \leq i \leq 3)$. Hence there is a triad $T' \subseteq P_1 \cup P_2 \cup P_3 \cup K$ with feet v_1, v_2, v_3. Since each P_i contains only one member of S' and K contains none, it follows that $|S' \cap V(T')| \leq 3$. But $V(T') \subseteq V(T) \cup B_n$, and so $S' \subseteq S \subseteq V(T')$; and therefore $|S'| \leq 3$. This proves our claim that $|S'| \leq 3$.

Let $A = V(G) - V(K)$, $B = V(K) \cup S'$. Then (A,B) is a (≤ 3)-separation of G with $v_1, v_2, v_3 \in A$, and with

$$|V(G) - A| = |V(K)| \geq 1.$$

From the hypothesis, $|A| \leq 3$, and so since $v_1, v_2, v_3 \in A$ it follows that $A = \{v_1, v_2, v_3\}$. But for $1 \leq i < n$,

$$B_i \cap B = B_i \cap (V(K) \cup S') \subseteq B_i \cap (B_n \cup V(T)) = \emptyset$$

and $v_1, v_2, v_3 \notin B_i$, and so $B_i = \emptyset$ which is impossible; and therefore $n = 1$ as required. ∎

Let v_1, v_2, v_3 be mutually adjacent vertices of a graph G. We say G is *triangular* with respect to v_1, v_2, v_3 if G is simple, and either

(i) for some i $(1 \leq i \leq 3)$, $G \setminus v_i$ has maximum valency ≤ 2, and either $G \setminus v_i$ is a circuit or it has no circuit, or

(ii) all vertices of G have valency ≤ 3, there is at most one 3-valent vertex $v \neq v_1, v_2, v_3$, and $G \setminus \{v_1, v_2, v_3\}$ has no circuit, or

(iii) all vertices of G have valency ≤ 3, there is a triangle C with $v_1, v_2, v_3 \notin V(C)$, every 3-valent vertex of G is in $\{v_1, v_2, v_3\} \cup V(C)$, and every circuit of C except these two triangles meets both $\{v_1, v_2, v_3\}$ and $V(C)$.

The motivation for this is the following.

(3.2) *Let $v_1, v_2, v_3 \in V(G)$ be distinct, mutually adjacent vertices of G, and let T be a lean triad in G with feet v_1, v_2, v_3. Let $v_1, v_2, v_3 \in Z \subseteq V(T)$; then $G \mid Z$ is triangular with respect to v_1, v_2, v_3.*

Proof. It suffices to show that $G \mid V(T)$ is triangular. Let a be the 3-valent vertex of T, and for $1 \leq i \leq 3$ let P_i be the path of T between a and v_i. Let $K = G|\{v_1, v_2, v_3\}$. If $G|V(T) = T \cup K$ then (ii) holds as required, and so we may assume that there exist $u, v \in V(T)$, adjacent in G but not in $T \cup K$. Suppose first that $u, v \notin \{v_1, v_2, v_3\}$. As in (2) in (3.1), it follows that u, v are both adjacent in T to the 3-valent vertex a of T, and $G \mid V(T)$ has no other edge not in $T \cup K$. But then (iii) holds if $\{u, v, a\} \subseteq Z$, and (ii) holds otherwise. We may therefore assume that $u = v_1$ say. Since u, v are not adjacent in $T \cup K$ it follows that $v \notin \{v_1, v_2, v_3\}$, and $v \in V(P_2)$ say. Then $|E(P_1)| = 1$, and $G|V(P_2 \cup P_3)$ is a circuit, and so (i) holds. ∎

Let v_1, v_2, v_3 be distinct vertices of a graph G. By a *tripod* on v_1, v_2, v_3 we mean a subgraph $P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3$ of G consisting of

(i) two vertices a, b so that a, b, v_1, v_2, v_3 are all distinct

(ii) three paths P_1, P_2, P_3 of G between a and b, mutually disjoint except for a and b, and each with at least one internal vertex, and

(iii) three paths Q_1, Q_2, Q_3 of G, mutually disjoint, such that for $i = 1, 2, 3$, Q_i has ends u_i and v_i, where $u_i \in V(P_i) - \{a, b\}$, and no vertex of Q_i except u_i belongs to $V(P_1 \cup P_2 \cup P_3)$. (It is permitted that $u_i = v_i$ and hence $E(Q_i) = \emptyset$.)

We call Q_1, Q_2, Q_3 the *legs* of the tripod.

(3.3) *Let $Z \subseteq V(G)$ such that there is no 3-separation (A,B) of G with $Z \subseteq A$ and $|B-A| \geq 2$. Let H_0 be a tripod in G with feet $v_1,v_2,v_3 \in Z$ and with no other vertex in Z. Then there is a tripod H with feet v_1,v_2,v_3 and with no other vertex in Z, such that every leg of H is a subpath of a leg of H_0, and there is a path from $V(H)$ to Z disjoint from all the legs of H.*

Proof. Let $H = P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2 \cup Q_3$ (with the usual notation) be a tripod in G with feet v_1,v_2,v_3 and with no other vertex in Z, chosen with $Q_1 \cup Q_2 \cup Q_3$ minimal. Let the ends of P_1,P_2,P_3 be a,b. From the hypothesis, there is a path P of G from $V(P_1 \cup P_2 \cup P_3)$ to $Z \cup V(Q_1 \cup Q_2 \cup Q_3)$ with no vertex in $\{u_1,u_2,u_3\}$, where Q_i has ends u_i,v_i $(1 \leq i \leq 3)$. Choose a minimal such path P with ends $x \in V(P_1 \cup P_2 \cup P_3)$ and $y \in Z \cup V(Q_1 \cup Q_2 \cup Q_3)$. We may assume from the symmetry that x and a belong to the same component of $P_1 \cup P_2 \cup P_3 \setminus \{u_1,u_2,u_3\}$. Suppose that $y \in V(Q_1)$. Let P' be the subpath of P_1 between x and u_1 if $x \in V(P_1)$, or between a and u_1 if $x \notin V(P_1)$. Let H' be the tripod obtained from $H \cup P$ by deleting the edges and internal vertices of P'; then H' contradicts the choice of H. Consequently, $y \notin V(Q_1)$ and $y \notin V(Q_2), V(Q_3)$ similarly; and so $y \in Z - \{v_1,v_2,v_3\}$, as required. ∎

A tripod is *legless* if all its legs have no edges.

(3.4) *Let $v_1,v_2,v_3 \in V(G)$ be distinct, so that there is a tripod on v_1,v_2,v_3. If there is no 3-separation (A,B) with $v_1,v_2,v_3 \in A$, $|A| \geq 4$ and $|B-A| \geq 2$, then there is a legless tripod on v_1,v_2,v_3.*

Proof. Let H be a tripod on v_1,v_2,v_3 with legs Q_1,Q_2,Q_3, chosen with $Q_1 \cup Q_2 \cup Q_3$ minimal. Suppose that $|E(Q_1)| \neq \emptyset$, and let v_1' be the neighbour of v_1 in Q_1. Let $Z = \{v_1,v_2,v_3,v_1'\}$. Then $H \setminus v_1$ is a tripod on v_1',v_2,v_3, and so by (3.3) we may assume there is a path P of G between $V(H \setminus v_1)$ and Z, disjoint from the legs of $H \setminus v_1$. Consequently, v_1 is an end of P, and as in (3.3) we may choose another tripod in $H \cup P$ contradicting the choice of H. The result follows. ∎

The following follows from [12, theorem (2.4)], and we omit the proof, which is similar to that of (2.4).

(3.5) *Let $v_1,v_2,v_3 \in V(G)$ be distinct, such that there is no (≤ 2)-separation (A,B) of G with $v_1,v_2,v_3 \in A$ and $|B-A| \geq 2$. Then either G contains a tripod on v_1,v_2,v_3, or G can be drawn in a disc with v_1,v_2,v_3 on the boundary.*

From (3.1), (3.4), (3.5) we deduce:

(3.6) *Let v_1,v_2,v_3 be mutually adjacent vertices of a 4-connected simple non-planar graph G. Let $Z \subseteq V(G)$ with $v_1,v_2,v_3 \in Z$ such that $G|Z$ is not triangular. Then there is a 5-cluster*

$$\{\{v_1\},\{v_2\},\{v_3\},X_1,X_2\}$$

in G such that $Z \cap X_1, Z \cap X_2 \neq \emptyset$.

Proof. Since G is non-planar, it cannot be drawn in a disc with v_1,v_2,v_3 on the boundary, and since G is 3-connected it follows from (3.5) that there is a tripod on v_1,v_2,v_3. By (3.4) such a tripod can be chosen legless. Consequently, there are two triads T_1,T_2 on v_1,v_2,v_3, vertex-disjoint except for $\{v_1,v_2,v_3\}$. By (3.1) we may assume that for $i=1,2,T_i$ is lean and there is only one T_i-flap. Consequently, we may choose $T_1,\ldots,T_n$ with $n \geq 2$ maximum, such that

(1) *$T_1,\dots,T_n$ are lean triads on v_1,v_2,v_3, mutually vertex-disjoint except for v_1,v_2,v_3, such that for each i there is only one T_i-flap.*

We deduce:

(2) *For $1 \le i \le n, Z \not\subseteq V(T_i)$.*

As $G|Z$ is not triangular, this follows from (3.2).

(3) *If $Z \cap V(T_i) \ne \{v_1,v_2,v_3\}$ for some i then the theorem is true.*

For let $Z \cap V(T_1) \ne \{v_1,v_2,v_3\}$, say. Let $X_1 = V(T_1) - \{v_1,v_2,v_3\}$ and $X_2 = V(G) - V(T_1)$. Since there is only one T_1-flap, X_2 is a fragment. and $Z \cap X_2 \ne \emptyset$ by (2). Thus $\{\{v_1\},\{v_2\},\{v_3\},X_1,X_2\}$ satisfies the theorem. This proves (3).

We may assume therefore that $Z \cap V(T_i) = \{v_1,v_2,v_3\}$ for $1 \le i \le n$. Let $H = G \setminus \{v_1,v_2,v_3\}$ and $S_i = T_i \setminus \{v_1,v_2,v_3\}$ $(1 \le i \le n)$. Then H is connected, and $S_1,\dots,S_n$ are mutually disjoint non-null connected subgraphs of it.

(4) *If there exist distinct j,j' with $1 \le j,j' \le n$, and two disjoint paths P,P' of H such that*

- (i) *P has one end in Z, the other end in $V(S_j)$, and no internal vertex in S_i for any i, and*
- (ii) *P' has one end in Z, the other end in $V(S_{j'})$, and no internal vertex in S_i for any i*

then the theorem holds.

For $P \cup S_j$ and $P' \cup S_{j'}$ are disjoint connected subgraphs of H, and so there exist disjoint fragments X_1,X_2 of H with $V(P \cup S_j) \subseteq X_1$ and $V(P' \cup S_{j'}) \subseteq X_2$, with $X_1 \cup X_2$ maximal. Since H is connected it follows that X_1X_2 are adjacent, and so $\{\{v_1\},\{v_2\},\{v_3\},X_1,X_2\}$ satisfies the theorem. This proves (4).

We assume therefore that there do not exist P,P' as in (4). By Menger's theorem applied to the graph obtained from H by contracting all the edges of each S_i, there is a separation (X,Y) of H with $V(S_1\cup\dots\cup S_n) \subseteq X$ and $Z \cap V(H) \subseteq Y$, so that either $|X \cap Y| \le 1$ or $X \cap Y = V(S_j)$ for some j. The latter is impossible since $H \setminus V(S_j)$ is connected, $n \ge 2$ and $Z \cap V(H) \ne \emptyset$; and so $|X \cap Y| \le 1$. Since $|Z| \ge 5$ (because $G \mid Z$ is not triangular) and hence $|Z \cap V(H)| \ge 2$, we deduce that $|Y| \ge 2$ and $|X| \ge |V(S_1\cup\dots\cup S_n)| \ge n \ge 2$. But H is connected, and so $|X \cap Y| = 1, X \cap Y = \{u\}$, say; and $H|X$ is connected. Let T_{n+1} be a triad in G with feet v_1,v_2,v_3 and with $V(T_{n+1}) \subseteq (Y - \{u\}) \cup \{v_1,v_2,v_3\}$; this exists since G is 4-connected and $|Y| \ge 2$. By (3.1) we may choose T_{n+1} lean and so that there is only one T_{n+1}-flap in G, because $H|X$ is connected. But then $T_1,\dots,T_{n+1}$ contradict the choice of n. The result follows. ∎

(3.7) *Let G be a 5-connected simple non-apex graph with no K_6-minor, let $w \in V(G)$, and let Z be the set of all neighbours of w. Let $v_1,v_2,v_3 \in Z$ be distinct and mutually adjacent. Then $G \mid Z$ is triangular with respect to v_1,v_2,v_3. In particular, if G is 6-connected then w belongs to ≤ 2 K_4-subgraphs of G.*

Proof. Suppose that $G \mid Z$ is not triangular. By (3.6) applied to the 4-connected non-planar graph $G \setminus w$, there is a 5-cluster

$$\{\{v_1\},\{v_2\},\{v_3\},X_1,X_2\}$$

in $G \setminus w$ such that $Z \cap X_1$, $Z \cap X_2 \neq \emptyset$. But then

$$\{\{w\}, \{v_1\}, \{v_2\}, \{v_3\}, X_1, X_2\}$$

is a 6-cluster in G, a contradiction. Thus $G \mid Z$ is triangular with respect to v_1, v_2, v_3. Now suppose that G is 6-connected. By (2.7), no edge of G is in ≥ 4 triangles, and so $G \mid Z$ has maximum valency ≤ 3. Hence (ii) or (iii) holds in the definition of "triangular". If $G \mid Z$ has ≥ 3 triangles, then (ii) holds and $G \mid Z$ is isomorphic to K_4, contrary to (2.8). Thus $G \mid Z$ has ≤ 2 triangles, as required. ∎

The relevance of (3.7) to our problem about Hadwiger's conjecture derives from the following result of Mader [9]; it will often be used in the remainder of the paper without explicit reference.

(3.8) *Every Hadwiger graph is 6-connected.*

4. Nearly-disjoint K_4's

Let us say that $X \subseteq V(G)$ is a *4-clique* if $|X| = 4$ and every two vertices of X are adjacent. A consequence of (3.7) and (3.8) is that in every non-apex Hadwiger graph, every vertex is in at most two 4-cliques. In this section we prove a complementary result, that there do not exist three 4-cliques pairwise meeting in ≤ 2 vertices.

First we need the following lemma.

(4.1) *Let $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3$ be distinct vertices of a 6-connected simple graph G, such that $\{x_1, y_1, z_2, z_3\}$, $\{x_2, y_2, z_3, z_1\}$, $\{x_3, y_3, z_1, z_2\}$ are 4-cliques. Suppose, moreover, that there is a partition X, Y of $V(G) - \{z_1, z_2, z_3\}$ with $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$, such that x_1y_1, x_2y_2, x_3y_3 are the only edges of G with one end in X and the other in Y. Then G has a K_6-minor.*

Proof. Let $Z = \{z_1, z_2, z_3\}$.

(1) *X, Y are fragments, and we may assume that $|X|, |Y| \geq 4$.*

For if $G \mid X$ (say) is not connected, let D be a component of $G \mid X$ with $x_3 \notin V(D)$. Then $(V(D) \cup \{y_1, y_2, z_1, z_2, z_3\}, V(G) - V(D))$ is a 5-separation of G, contradicting that G is 6-connected. Thus X, Y are fragments. If $|X| \leq 3$ say, then $X = \{x_1, x_2, x_3\}$. Since x_1 has valency ≥ 6 it is adjacent to x_2, x_3 and to every member of Z, and similarly for x_2, x_3; but then $Z \cup \{x_1, x_2, x_3\}$ is a 6-clique, as required.

Let f_1, f_2, f_3 be the edges with ends z_2z_3, z_3z_1 and z_1z_2 respectively.

(2) *$G \setminus \{f_1, f_2, f_3\}$ is not planar.*

For $|E(G)| \geq 3|V(G)|$ since G is 6-connected, and so $|E(G \setminus \{f_1, f_2, f_3\})| \geq 3|V(G)| - 3$: and (2) follows.

Let C be the circuit of G formed by the six vertices $z_1, x_2, z_3, x_1, z_2, x_3$ in that order. Let

$$H = G \mid (X \cup Z) \setminus (\{f_1, f_2, f_3\} \cup E(C)).$$

From (2), we may assume by exchanging X and Y that H cannot be drawn in a disc with $V(C)$ on the boundary in order. There is no (≤ 3)-separation (A, B) of H with $V(C) \subseteq A \neq V(H)$, and so from [12, theorem (2.4)] (or from (2.4)), we deduce

(3) *There are disjoint paths P, Q of H with ends p_1, p_2 and q_1, q_2 respectively, so that $p_1, q_1, p_2, q_2 \in V(C)$ and occur in C in that order, and no other vertices of P or Q lie in C.*

(The requirement that "no other vertices of P or Q lie in C" is satisfied by choosing P and Q with $P \cup Q$ minimal.) Next, we claim

(4) *If P, Q can be chosen with*

$$\{p_1, p_2\} \cap \{x_1, x_2, x_3\} \neq \emptyset \quad \textit{and} \quad \{q_1, q_2\} \cap \{x_1, x_2, x_3\} \neq \emptyset$$

then G has a K_6-minor.

For if so, we may assume that $p_1 = x_1$ and $q_1 = x_2$. Then $p_2 \neq z_2, z_3$, and so $p_2 \in \{z_1, x_3\}$, and similarly $q_2 \in \{z_2, x_3\}$. Choose disjoint fragments A, B of $G \mid X$ with $V(P) - Z \subseteq A$ and $V(Q) - Z \subseteq B$, with $A \cup B$ maximal. Since X is a fragment by (1), it follows that AB are adjacent, and so by (1) again,

$$\{\{z_1\}, \{z_2\}, \{z_3\}, A, B, Y\}$$

is a 6-cluster in G as required.

From (4), we may therefore assume that $p_1 = z_1, p_2 = z_2$, $q_1 = x_3$, and $q_2 \in \{x_1, z_3, x_2\}$.

(5) *There are two disjoint paths of $H \setminus \{z_1, z_2\}$ from $\{x_1, z_3, x_2\}$ to $V(P) \cup \{x_3\}$.*

For if not, there is a (≤ 3)-separation (A, B) of H with $Z \cup \{x_1, x_2\} \subseteq A$ and $V(P) \cup \{x_3\} \subseteq B$. Then $(A \cup Y \cup \{x_3\}, B)$ is a (≤ 4)-separation of G, and $B \neq V(G)$, and so $A \cup Y \cup \{x_3\} = V(G)$ since G is 6-connected; that is, $V(H) = A \cup \{x_3\}$. Since $V(P) \subseteq B - \{x_3\} \subseteq A \cap B$ and $|V(P)| \geq 3 \geq |A \cap B|$ it follows that $V(P) = A \cap B$, and so $V(Q) \cap A \cap B = \emptyset$. Yet Q has one end in A and the other in B, a contradiction. The claim follows.

From (5) and the existence of Q, we deduce that there are two disjoint paths of $H \setminus \{z_1, z_2\}$ from $\{x_1, z_3, x_2\}$ to $V(P) \cup \{x_3\}$, both with no internal vertex in P, and one ending at x_3, which we may as well choose Q to be. In other words, we may assume that there is a path R of H from $\{x_1, z_3, x_2\}$ to some $r \in V(P) - \{z_1, z_2\}$, with no vertex in P except r, with only one vertex in $\{x_1, z_3, x_2\}$, and with no vertex in Q. If R has x_1 or x_2 as one end then we may choose P, Q to satisfy (4). Thus we may assume that R has ends z_3, r; and Q has ends x_1, x_3, from the symmetry between x_1 and x_2.

(6) *We may assume that there is a path S of H from x_2 to some $s \in V(Q) - \{x_1, x_3\}$ with no vertex in Q except s, and disjoint from $P \cup R$.*

For let D be the component of $G \setminus V(C)$ containing r. Since G is 6-connected, every vertex of C has a neighbour in $V(D)$, and so there is a path of H from x_2 to r and hence to $V(P \cup Q \cup R) - V(C)$ with no vertex in C except x_2. Let S be a minimal such path, with ends x_2, s say. Then $s \in V(P \cup Q \cup R) - V(C)$, and no vertex of S except s is in $V(P \cup Q \cup R)$. If $s \in V(P \cup R)$ then P and Q can be chosen as in (4). We may therefore assume that $s \in V(Q)$. This proves (6).

Let B be the component of $H \backslash V(C \cup Q \cup S)$ which contains r. The only vertices of G not in B which have a neighbour in B are in $V(C) \cup V(Q \cup S)$, and there are ≥ 6 such vertices since G is 6-connected. Since

$$|V(C) - V(Q \cup S)| = 3$$

at least three of them are in $Q \cup S$. We may therefore assume from the symmetry between x_1, x_2 and x_3, that there are two vertices u, v in Q with a neighbour in B, and v lies in the component of $Q \setminus s$ containing x_3, and u lies in Q between v and x_1 (possibly $u = x_1$). Let A be the component of $Q \backslash v$ containing u, s and x_1. Then by (1),

$$\{\{z_1\}, \{z_2\}, \{z_3\}, V(B), V(A \cup S), (V(Q) - V(A)) \cup Y\}$$

is a 6-cluster in G, and so G has a K_6-minor, as required. ∎

Secondly, we need Mader's "H-Wege" theorem [8], the following. We say $S \subseteq V(G)$ is *stable* if no edge has both ends in S.

(4.2) *Let G be a graph, let $S \subseteq V(G)$ be stable, and let $k \geq 0$ be an integer. Then exactly one of the following holds:*

- (i) *there are k paths of G, each with distinct ends both in S, such that each $v \in V(G) - S$ is in at most one of the paths*
- (ii) *there exist $W \subseteq V(G) - S$ and a partition $Y_1, \ldots, Y_n$ of $V(G) - (S \cup W)$, and for $1 \leq i \leq n$ a subset $X_i \subseteq Y_i$, such that*
  - (a) $|W| + \sum_{1 \leq i \leq n} \left\lfloor \frac{1}{2} |X_i| \right\rfloor < k$,
  - (b) *no vertex in $Y_i - X_i$ has a neighbour in $V(G) - (W \cup Y_i)$*
  - (c) *every path of $G \setminus W$ with distinct ends both in S has an edge with both ends in Y_i for some i.*

Let $L_1, \ldots, L_t$ be subsets of $V(G)$, where G is a graph. A path P of G with ends u, v is *good* if there exist distinct i, j with $1 \leq i, j \leq t$ such that $u \in L_i$ and $v \in L_j$. From (4.2) we deduce:

(4.3) *Let G be a graph, let $L_1, \ldots, L_t$ be subsets of $V(G)$, and let $k \geq 0$ be an integer. Then exactly one of the following holds:*

- (i) *there are k good paths of G, mutually vertex-disjoint*
- (ii) *there exists $W \subseteq V(G)$ and a partition $Y_1, \ldots, Y_n$ of $V(G) - W$, and for $1 \leq i \leq n$ a subset $X_i \subseteq Y_i$, such that*
  - (a) $|W| + \sum_{1 \leq i \leq n} \left\lfloor \frac{1}{2} |X_i| \right\rfloor < k$
  - (b) *for $1 \leq i \leq n$, no vertex in $Y_i - X_i$ has a neighbour in $V(G) - (W \cup Y_i)$, and $Y_i \cap L_j \subseteq X_i$ for $1 \leq j \leq t$*

(c) *every good path P in G with $V(P) \cap W = \emptyset$ has an edge with both ends in Y_i for some i.*

Proof. For $1 \leq i \leq t$ let s_i be a new vertex, and add $s_1,\ldots,s_t$ to G, making s_i adjacent to all vertices in L_i ($1 \leq i \leq t$). Let $S = \{s_1,\ldots,s_t\}$, and let the graph we construct be G'. Then (4.3) follows by (4.2) applied to G', S. ∎

We use (4.3) to prove the following.

(4.4) *Let G be a simple, 6-connected non-apex graph with no K_6-minor. Then there do not exist three 4-cliques L_1, L_2, L_3 of G such that $|L_i \cap L_j| \leq 2$ ($1 \leq i < j \leq 3$).*

Proof. Suppose that such L_1, L_2, L_3 exist, and choose them with $|L_1 \cup L_2 \cup L_3|$ minimum. By (2.7), $|L_i \cap L_j| \leq 1$ for $1 \leq i < j \leq 3$, and by (3.7) $L_1 \cap L_2 \cap L_3 = \emptyset$. Define "good" as before.

(1) *There do not exist 6 mutually disjoint good paths in G.*

For suppose such paths exist, $P_1,\ldots,P_6$ say. For $1 \leq i < i' \leq 6$, $V(P_i)$ meets ≥ 2 of L_1, L_2, L_3, and so does $V(P_{i'})$, and so there exists j with $1 \leq j \leq 3$ such that $V(P_i) \cap L_j \neq \emptyset \neq V(P_{i'}) \cap L_j$. Consequently, a vertex of P_i is adjacent to a vertex of $P_{i'}$, since $G \mid L_j$ is complete. Hence $\{V(P_1),\ldots,V(P_6)\}$ is a 6-cluster in G, a contradiction.

From (4.3) we deduce

(2) *There exists $W \subseteq V(G)$ and a partition $Y_1,\ldots,Y_n$ of $V(G) - W$ (we permit $Y_i = \emptyset$), and for $1 \leq i \leq n$ a subset $X_i \subseteq Y_i$, such that*

(a) $|W| + \sum_{1 \leq i \leq n} \left\lfloor \frac{1}{2}|X_i| \right\rfloor \leq 5$

(b) *for $1 \leq i \leq n$, no vertex in $Y_i - X_i$ has a neighbour in $V(G) - (W \cup Y_i)$, and $Y_i \cap (L_1 \cup L_2 \cup L_3) \subseteq X_i$*

(iii) every good path disjoint from W has an edge with both ends in Y_i for some i.

Choose W and $Y_1,\ldots,Y_n, X_1,\ldots,X_n$ as in (2) with W maximal. We may assume that $Y_i \neq \emptyset$ for each i since otherwise Y_i may be omitted.

Define $M = (L_1 \cap L_2) \cup (L_2 \cap L_3) \cup (L_3 \cap L_1)$. Then $|M| \leq 3$, from the hypothesis. If $v \in M$, then v forms a 1-vertex good path, and so $v \in W$ by (2)(c). Consequently,

(3) $M \subseteq W$.

We claim:

(4) $n \geq 2$.

For $L_1 \cup L_2 \cup L_3 \subseteq W \cup X_1 \cup \ldots \cup X_n$ and $|L_1 \cup L_2 \cup L_3| = 12 - |M| \geq 9$, and $|W| \leq 5$ by (2)(a). Thus, $n \geq 1$. Suppose that $n = 1$. Then

$$|W| + \left\lfloor \frac{1}{2}|X_1| \right\rfloor \leq 5,$$

but

$$|W| + |X_1| \geq |L_1 \cup L_2 \cup L_3| = 12 - |M| \geq 12 - |W|,$$

by (3), and so

$$10 \geq 2(|W| + \left\lfloor \frac{1}{2}|X_1| \right\rfloor) \geq 2|W| + |X_1| - 1 \geq 11,$$

a contradiction. Thus $n \geq 2$.

(5) *For $1 \leq i \leq n, |X_i|$ is odd.*

For suppose that $|X_1|$ is even, say. If $X_1 \neq \emptyset$, let $v \in X_1$, let $W' = W \cup \{v\}$, $X_1' = X_1 - \{v\}$, $Y_1' = Y_1 - \{v\}$, and $X_i' = X_i, Y_i' = Y_i$ for $2 \leq i \leq n$; then $W', X_1', \ldots, X_n', Y_1', \ldots, Y_n'$ satisfy (2), contrary to the maximality of W. Hence $X_1 = \emptyset$, and so $(Y_1 \cup W, Y_2 \cup \ldots \cup Y_n \cup W)$ is a separation of G. But $n \geq 2$ by (4), and $Y_1, Y_2 \neq \emptyset$, and so $Y_1 \cup W \neq V(G)$ and $Y_2 \cup \ldots \cup Y_n \cup W \neq V(G)$. Since G is 6-connected it follows that $|W| \geq 6$, contrary to (2)(a). This proves (5).

For $1 \leq i \leq 3$, let Z_i be the union of the vertex sets of all paths P with $V(P) \cap W = \emptyset$ such that P has no edge with both ends in Y_j for $1 \leq j \leq n$, and $V(P) \cap L_i \neq \emptyset$.

(6) *For $1 \leq i \leq 3$, $L_i - W \subseteq Z_i \subseteq V(G) - W$, and Z_1, Z_2, Z_3 are mutually disjoint.*

The first claim is immediate, and the second follows from (2)(c).

(7) *For $1 \leq i \leq 3$, $Z_i \subseteq X_1 \cup \ldots \cup X_n$.*

For suppose that $v \in Z_i \cap (Y_j - X_j)$ for some j with $1 \leq j \leq n$. Let P be a path of $G \setminus W$ from v to L_i such that for $1 \leq j \leq n$, no edge of P has both ends in Y_j. Since $V(P) \neq \{v\}$ (because $v \notin L_i$), there is an edge e of P incident with v. By (2)(b), both ends of e are in Y_j, contrary to the choice of P. The claim follows.

(8) *For $1 \leq i, i' \leq 3$ with $i \neq i'$, every path of $G \setminus W$ from Z_i to $Z_{i'}$ has ≥ 2 vertices in X_j for some j.*

Let Q be a path of $G \setminus W$ from $v \in Z_1$ to $w \in Z_2$, say. Let P be a path of $G \setminus W$ from $u \in L_1$ to v, and let R be a path of $G \setminus W$ from w to $x \in L_2$, such that P and R both have no edge with both ends in Y_j for any j $(1 \leq j \leq n)$. Let $S \subseteq P \cup Q \cup R$ be a path from u to x. Then S is good, and so there exists $e \in E(S)$ with both ends in Y_j for some j. By the choice of P and R, $e \notin E(P) \cup E(R)$, and so $e \in E(Q)$. Hence Q has ≥ 2 vertices in Y_j. But Q has ends v, w, and $v \in Z_1 \subseteq X_1 \cup \ldots \cup X_n$ and $w \in Z_2 \subseteq X_1 \cup \ldots \cup X_n$ by (7). Thus by (2)(b), Q has at least two vertices in X_j, as required.

(9) *For $1 \leq i \leq 3$, $|Z_i| \leq 5 - |W|$.*

For suppose that $|Z_1| \geq 6 - |W|$, say. Now $|L_2 \cup L_3| \geq 7$, and so

$$|L_2 \cup L_3 - W| \geq 7 - |W| \geq 6 - |W|.$$

But $G\backslash W$ is $(6-|W|)$-connected, and so there are $6-|W|$ paths P_i $(1\leq i\leq 6-|W|)$ of $G\backslash W$ from Z_1 to $L_2\cup L_3-W$, mutually disjoint. By (8) each P_i has two vertices in some X_j, and so

$$\sum_{1\leq j\leq n}\left\lfloor\frac{1}{2}|X_j|\right\rfloor\geq 6-|W|,$$

contrary to (2)(a).

(10) $|W|\leq 3$.

For by (9) and (6),

$$12=\sum_{1\leq i\leq 3}|L_i|\leq\sum_{1\leq i\leq 3}(|Z_i|+|L_i\cap W|)\leq 3(5-|W|)+2|W|=15-|W|.$$

The claim follows.

Let $Z_0=V(G)-(W\cup Z_1\cup Z_2\cup Z_3)$. Then Z_0,Z_1,Z_2,Z_3,W is a partition of $V(G)$.

(11) *If $u,v\in V(G)-W$ are adjacent, then either $u,v\in Z_i$ for some i $(0\leq i\leq 3)$ or $u,v\in Y_j$ for some j $(1\leq j\leq n)$.*

For suppose that $u\in Z_1\cap Y_1$ and $v\in Y_2$, say, and $e\in E(G)$ has ends u,v. Then e does not have both ends in Y_j for $1\leq j\leq n$, and so $v\in Z_1$ (since $u\in Z_1$) by definition of Z_1, as required.

(12) *For $1\leq j\leq n$, if $|W\cup X_j|\leq 5$ then $X_j=Y_j$.*

For suppose that $X_j\neq Y_j$. Since $(W\cup Y_j, V(G)-(Y_j-X_j))$ is a separation of G and $V(G)-(Y_j-X_j)\neq V(G)$ (since $X_j\neq Y_j$) and $W\cup Y_j\neq V(G)$ (since $n\geq 2$ by (4)) and G is 6-connected, it follows that $|W\cup X_j|\geq 6$, as required.

(13) $|X_j|\geq 3$ *for* $1\leq j\leq n$.

Reorder the indices so that $|X_j|\geq 3$ for $1\leq j\leq m$ and $|X_j|=1$ for $m<j\leq n$. By (10) and (12), $X_j=Y_j$ for $m<j\leq n$. Let $U=X_{m+1}\cup\ldots\cup X_n$, and suppose that $0\leq i\leq 3$ and $Z_i\cap U\neq\emptyset$. Let N be the set of vertices in $V(G)-(Z_i\cap U)$ with a neighbour in $Z_i\cap U$. If $v\in N-(W\cup Z_i)$, let v be adjacent to $u\in Z_i\cap U$; by (11) there exists j with $1\leq j\leq n$ such that $u,v\in Y_j$, and so $|Y_j|\geq 2$ and hence $j\leq m$, contradicting that $u\in U$. There is therefore no such v, and so $N\subseteq W\cup Z_i$. Now for all i' with $1\leq i'\leq 3$,

$$\emptyset\neq V(L_{i'})-W\subseteq Z_{i'}$$

by (6) and (10); and consequently $W\cup Z_i\neq V(G)$. Since G is 6-connected, it follows that $|N|\geq 6$, and so $i=0$ by (9). In particular, N and $Z_1\cup Z_2\cup Z_3$ are disjoint subsets of $W\cup X_1\cup\ldots\cup X_m$. Consequently

$$|N|+\sum_{1\leq i\leq 3}|Z_i|\leq|W|+\sum_{1\leq j\leq m}|X_j|.$$

But $|N|\geq 6$,

$$\sum_{1\leq i\leq 3}|Z_i|\geq\sum_{1\leq i\leq 3}|L_i-W|\geq 12-2|W|,$$

and by (2),

$$\sum_{1\le j\le m} |X_j| \le 3 \sum_{1\le j\le m} \left\lfloor \frac{1}{2}|X_j| \right\rfloor \le 3(5-|W|);$$

and so

$$6+(12-2|W|) \le |W| + (15-3|W|),$$

a contradiction. This proves (13).

(14) $|X_1 \cup \ldots \cup X_n - (L_1 \cup L_2 \cup L_3 - W)| \le 3 + |M| - 2|W|$, *with strict inequality if* $|X_j| > 3$ *for some* j.

For let $s = |X_1 \cup \ldots \cup X_n - (L_1 \cup L_2 \cup L_3 - W)|$. Then

$$|X_1 \cup \ldots \cup X_n| \ge s + 12 - |M| - |W|.$$

But $|X_j| \le 3 \left\lfloor \frac{1}{2}|X_j| \right\rfloor$ for $1 \le j \le n$, and so

$$3 \sum_{1\le j\le n} \left\lfloor \frac{1}{2}|X_j| \right\rfloor \ge \sum_{1\le j\le n} |X_j| \ge s + 12 - |M| - |W|,$$

with strict inequality if $|X_j| > 3$ for some j. From (2)(a), we deduce that

$$3(5-|W|) \ge s + 12 - |M| - |W|,$$

that is, $s \le 3 + |M| - 2|W|$; and again with strict inequality if $|X_j| > 3$ for some j, as required.

(15) *For* $1 \le j \le n$ *and* $1 \le i \le 3$, $|Z_i \cap X_j| < \frac{1}{2}|X_j|$.

For suppose that $|Z_1 \cap X_1| \ge \frac{1}{2}|X_1|$. Since $X_1 \ne \emptyset$ by (5), there exists $v \in Z_1 \cap X_1$. Since $|L_2 \cup L_3 - W| \ge |L_2 \cup L_3| - |W| \ge 7 - |W|$, and $G \setminus W$ is $(6-|W|)$-connected, there are $6-|W|$ paths of $G \setminus W$ between Z_1 and $L_2 \cup L_3 - W$, disjoint except possibly for v. Choose them with no internal vertex in Z_1. Each has two vertices in X_j for some j, by (8); but at most

$$\sum_{2\le j\le n} \left\lfloor \frac{1}{2}|X_j| \right\rfloor \le 5 - |W| - \left\lfloor \frac{1}{2}|X_1| \right\rfloor$$

of them have two vertices in X_j for some $j \ne 1$. Thus at least $1 + \left\lfloor \frac{1}{2}|X_1| \right\rfloor$ of them have two vertices in X_1. But each has only one vertex in Z_1, and so has a vertex in X_1 which does not belong to Z_1; and all these vertices are different. Consequently,

$$|X_1 - Z_1| \ge 1 + \left\lfloor \frac{1}{2}|X_1| \right\rfloor$$

and the result follows.

(16) $|W| \leq 2$.

For suppose that $|W| \geq 3$. By (14), $3 + |M| - 2|W| \geq 0$ and by (3), $|W| \geq |M|$; and so $W = M$, and $|W| = 3$. By (13) and (14), $|X_j| = 3$ for all j, and

$$X_1 \cup \ldots \cup X_n = L_1 \cup L_2 \cup L_3 - W.$$

But $|L_1 \cup L_2 \cup L_3 - W| = 6$ since $|W| = 3$ and $W = M$, and so $n = 2$. For i = 1, 2, 3, by (15) and (7), $|Z_i \cap X_1| = 1$ and $|Z_i \cap X_2| = 1$, and so $|Z_i| = 2$. Since $L_i - W \subseteq Z_i$ and $|L_1 \cup L_2 \cup L_3 - W| = 6$ it follows that $Z_i = L_i - W$ for $1 \leq i \leq 3$. This contradicts (4.1) (using (11)).

(17) *For* $1 \leq j \leq n$, *if* $|X_j| = 3$ *then* $Y_j = X_j$.

This follows from (12) since $|W \cup X_j| \leq 5$ by (16).

(18) *For* $1 \leq j \leq n$, *if* $|X_j| = 3$ *then* $X_j \cap Z_0 = \emptyset$.

For suppose that $|X_1| = 3$, say, and $v \in X_1 \cap Z_0$. By (17), $Y_1 = X_1$, and so by (11), all neighbours of v belong to $X_1 \cup W \cup (Z_0 \cap (X_2 \cup \ldots \cup X_n))$. But by (14), $|Z_0 \cap (X_1 \cup \ldots \cup X_n)| \leq |X_1 \cup \ldots \cup X_n - (L_1 \cup L_2 \cup L_3 - W)| \leq 3 + |M| - 2|W|$, and so v has at most

$$3 + |M| - 2|W| - |Z_0 \cap X_1|$$

neighbours not in $X_1 \cup W$; and hence it has $\leq 5 + |M| - |W| - |Z_0 \cap X_1|$ neighbours altogether. But $|Z_0 \cap X_1| \geq 0$ and $|M| \leq |W|$, and so v has valency ≤ 5, a contradiction. This proves (18).

(19) $|X_j| \geq 5$ *for* $1 \leq j \leq n$.

For suppose $|X_1| = 3$ say. By (18), $X_1 \cap Z_0 = \emptyset$, and so by (15), $|Z_i \cap X_1| = 1$ for $1 \leq i \leq 3$. By (17), $Y_1 = X_1$. Let $X_1 = \{v_1, v_2, v_3\}$ where $v_i \in Z_i$ $(1 \leq i \leq 3)$. Let $1 \leq i \leq 3$. By (11) every neighbour of v_i is in $W \cup X_1 \cup Z_i$; and by (9) $|Z_i| \leq 5 - |W|$. Consequently,

$$|W \cup X_1 \cup Z_i| \leq |W| + |Z_i| + |X_1 - Z_i| \leq 7.$$

Since v_i has valency ≥ 6, it follows that v_i is adjacent to every vertex in $W \cup X_1 \cup Z_i$ except v_i. Suppose for a contradiction that $w \in W$, and let $L_0 = X_1 \cup \{w\}$; then L_0 is a 4-clique. Now $L_0 \neq L_1, L_2, L_3$ since $L_0 - W \not\subseteq Z_1, Z_2, Z_3$, and so w belongs to at most two of L_0, L_1, L_2, L_3, by (3.7). Consequently, $w \notin M$, and so $M = \emptyset$. From the minimality of $L_1 \cup L_2 \cup L_3$, it follows that $L_0 \cap L_i = \emptyset$ for $1 \leq i \leq 3$, and so $X_1 \cap L_i = \emptyset$ for $1 \leq i \leq 3$, and $w \notin L_1 \cup L_2 \cup L_3$; indeed, $W \cap (L_1 \cup L_2 \cup L_3) = \emptyset$. Thus

$$\sum |X_i| \geq |X_1| + |L_1 \cup L_2 \cup L_3| = 15,$$

and so $\sum \left\lfloor \frac{1}{2}|X_i| \right\rfloor \geq 5$; yet $W \neq \emptyset$, contrary to (2)(a). It follows that $W = \emptyset$. Now for $1 \leq i \leq 3$, v_i is adjacent to every other vertex of $Z_i \cup X_1$, as we saw; and $|Z_i| = 5$. Hence $|Z_1 \cup Z_2 \cup Z_3| = 15$. But by (14),

$$|X_1 \cup \ldots \cup X_n| \leq 3 + |L_1 \cup L_2 \cup L_3| = 15$$

and so we have equality throughout. In particular $|X_1 \cup \ldots \cup X_n| = 15$, and each $|X_i| = 3$ since we have equality in (14). Hence $n = 5$. Since $L_1 \subseteq Z_1$ and $|L_1| = 4$, we may assume that $v_1 \notin L_1$, by the symmetry between $X_1, \ldots, X_5$; but then $G\,|\,(L_1 \cup \{v_1\})$ is isomorphic to K_5, contrary to (2.8). This proves (19).

(20) $n=2$.

For $n \geq 2$ by (4). But by (2)(a), $\sum \left\lfloor \frac{1}{2}|X_i| \right\rfloor < 6$, and so $n \leq 2$ by (19).

(21) $X_1 \cup X_2 = L_1 \cup L_2 \cup L_3 - W$, *and* $W = M$, *and* $|W| \leq 1$.

For let $s = |X_1 \cup X_2 - (L_1 \cup L_2 \cup L_3 - W)|$. As in (14),

$$|X_1 \cup X_2| \geq s + 12 - |M| - |W|.$$

By (19), for $j = 1,2$,

$$|X_j| \leq \frac{5}{2} \left\lfloor \frac{1}{2}|X_j| \right\rfloor,$$

and so from (2)(a),

$$\frac{5}{2}(5 - |W|) \geq \frac{5}{2} \left\lfloor \frac{1}{2}|X_1| \right\rfloor + \frac{5}{2} \left\lfloor \frac{1}{2}|X_2| \right\rfloor \geq |X_1 \cup X_2| \geq s + 12 - |M| - |W|,$$

that is,

$$2s \leq 1 - 2(|W| - |M|) - |W|.$$

Hence $s = 0$, and $|W| = |M| \leq 1$, as required.

(22) $W = \emptyset$.

For, if $W \neq \emptyset$, then by (21), $|W| = 1, |M| = 1$, and hence $|X_1 \cup X_2| = 10$ by (21) again. By (19), $|X_1| = |X_2| = 5$. Let $X_i = \{a_i, b_i, c_i, d_i, e_i\}$ $(i = 1,2)$, and $W = \{w\}$. Then by (15), we may assume that $L_1 = \{a_1, b_1, a_2, w\}$, $L_2 = \{c_1, b_2, c_2, w\}$, $L_3 = \{d_1, e_1, d_2, e_2\}$. Since G is 6-connected, $G \setminus \{w, a_2, c_1, d_1, e_1\}$ is connected; and since $Y_1 \cup Y_2 \cup \{w\} = V(G)$, there exists $u_1 \in Y_1 - \{c_1, d_1, e_1\}$ and $u_2 \in Y_2 - \{a_2\}$ so that $u_1 u_2$ are adjacent. By (2)(b), $u_1 \in X_1$ and $u_2 \in X_2$; hence $u_1 \in \{a_1, b_1\}$, and $u_2 \in \{b_2, c_2, d_2, e_2\}$. But this contradicts (11). Hence $W = \emptyset$, as required.

By (21) and (22), $|X_1 \cup X_2| = 12$, and so we may assume that $|X_1| = 5$ and $|X_2| = 7$. By (12) $Y_1 = X_1$. Let $X_1 = \{a_1, b_1, c_1, d_1, e_1\}$, $X_2 = \{a_2, b_2, c_2, d_2, e_2, f_2, g_2\}$. By (15) we may assume that $L_1 = \{a_1, b_1, a_2, b_2\}$, $L_2 = \{c_1, d_1, c_2, d_2\}$, $L_3 = \{e_1, e_2, f_2, g_2\}$. Now by (7), $Z_1 = \{a_1, b_1, a_2, b_2\}$, and so $|Z_1 \cup X_1| = 7$. Hence by (11), a_1, b_1 are both adjacent to every other vertex in X_1, and similarly so are c_1, d_1. But then $G \mid X_1$ is isomorphic to K_5, contrary to (2.8). ∎

Let us say two 4-cliques L_1, L_2 in G are *close* if $|L_1 \cap L_2| \geq 3$. Then we have

(4.5) *Let G be 6-connected, simple, and non-apex, with no K_6-minor. Then*

(i) *closeness is an equivalence relation on 4-cliques*

(ii) *each equivalence class has ≤ 2 members*

(iii) *there are ≤ 2 equivalence classes*

(iv) *there are ≤ 10 vertices in 4-cliques.*

Proof. (i) follows from (2.7), and (ii) from (3.7), and (iii) from (4.4). To deduce (iv), we see that from (ii), each equivalence class has ≤ 2 members and the union of its members has cardinality ≤ 5; and so (iv) follows from (iii). ∎

5. Vertices of valency 6

So far, our results have been about non-apex 6-connected graphs with no K_6-minor. However, now we need to use some further properties of Hadwiger graphs. We shall need the following throughout the paper.

(5.1) *Let G be a Hadwiger graph, and let $X_1,\ldots,X_k$ be disjoint fragments of G. Let $Z \subseteq X_1 \cup \ldots \cup X_k$ with $Z \neq \emptyset$ such that $X_i - Z$ is stable for $1 \leq i \leq k$. Then there is a 5-colouring ϕ of $G \setminus Z$ such that for $1 \leq i \leq k, \phi(x) = \phi(y)$ for all $x,y \in X_i - Z$, and such that for $1 \leq i < j \leq k$, if X_iX_j are adjacent then $\phi(x) \neq \phi(y)$ for $x \in X_i - Z$ and $y \in X_j - Z$.*

Proof. We may assume that $|X_i| \geq 2$ for some i, since otherwise the result is clear. Let H be obtained from G by contracting all edges of $G \mid X_i$ for $1 \leq i \leq k$. Since H is a loopless minor of G and $|V(H)| < |V(G)|$, there is a 5-colouring ψ of H. For $v \in V(G) - Z$, let u be the corresponding vertex of H, and define $\phi(v) = \psi(u)$; then ϕ satisfies (5.1). ∎

The first application of (5.1) is the following.

(5.2) *Let G be a Hadwiger graph, let $v \in V(G)$, and let N be the set of neighbours of v. Then $G \mid N$ has no stable set of cardinality $|N| - 3$.*

Proof. Suppose that $A \subseteq N$ is stable and $|A| = |N| - 3$, and choose $a \in A$. By (5.1) with $X_1 = A \cup \{v\}$, there is a 5-colouring ϕ of $G \setminus v$ such that $\phi(u) = \phi(a)$ for all $u \in A$. Choose $\alpha \in \{1,\ldots,5\}$ with $\alpha \neq \phi(a), \phi(b), \phi(c), \phi(d)$, where $N - A = \{b,c,d\}$. Then setting $\phi(v) = \alpha$ defines a 5-colouring of G, a contradiction. ∎

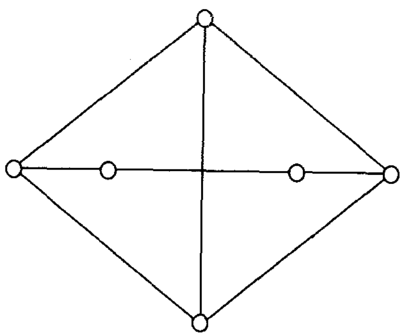

Fig. 1. A diamond

We call graphs isomorphic to the six-vertex graph shown in figure 1 *diamonds*. The next result was also proved by J. Mayer [10, 11].

(5.3) *Let v be a 6-valent vertex of a non-apex Hadwiger graph G, and let N be the set of neighbours of v. Then $G \mid N$ has exactly two triangles, and either $G \mid N$ is a diamond, or the two triangles are disjoint. In particular v belongs to exactly two 4-cliques, and every edge incident with v is in ≥ 2 triangles of G.*

Proof. Let $N = \{v_1,\ldots,v_6\}$. By (2.7) each edge of G is in ≤ 3 triangles, and so $G \mid N$ has maximum valency ≤ 3. By (5.2), $G \mid N$ has no stable set of cardinality

3, and hence it has a triangle by Ramsey's theorem, with vertex set $\{v_1,v_2,v_3\}$ say. Suppose first that some two of v_4,v_5,v_6 are not adjacent, say v_4v_5. Since $G \mid N$ has no 4-clique by (2.8) and no stable set of cardinality 3, we may assume that v_1v_4,v_2v_4 and v_3v_5 are adjacent. Since $G \mid N$ has maximum valency ≤ 3, $v_1v_5,v_1v_6,v_2v_5,v_2v_6,v_3v_4$ and v_3v_6 are non-adjacent. Hence v_4v_6 and v_5v_6 are adjacent and $G \mid N$ is a diamond. We may assume therefore that v_4v_5,v_5v_6 and v_4v_6 are all adjacent. Since $G \mid N$ has maximum valency ≤ 3 it has exactly two triangles and again the result is true. ∎

(5.4) *Let G be a non-apex Hadwiger graph, and let $u,v \in V(G)$ be adjacent, with the edge uv in ≥ 3 triangles. If u has valency 6 then v has valency ≥ 8.*

Proof. By (2.7), uv is in exactly three triangles; let the neighbours of u be x_1,x_2,x_3,v,u_1,u_2 where x_1,x_2,x_3 are adjacent to v. Since G has no K_5-subgraph by (2.8), we may assume that x_1x_2 are non-adjacent.

By (5.2), $\{x_1,x_2,x_3\}$ is not stable, and so x_3 is adjacent to x_1 or to x_2; and so we may assume x_2x_3 are adjacent. Since ux_3 is in ≤ 3 triangles by (2.7), not both u_1 and u_2 are adjacent to x_3, and so we may assume that u_1x_3 are non-adjacent. We suppose that v has valency ≤ 7. Let N be the set of two or three neighbours of v different from u,x_1,x_2,x_3.

(1) *$|N| = 3$ and each $y \in N$ is adjacent to one of x_1,x_2.*

For otherwise we may choose $A \subseteq \{x_1,x_2\} \cup N$, stable, with $x_1,x_2 \in A$ and with $|N - A| \leq 2$. By (5.1) with $X_1 = A \cup \{v\}$, $X_2 = \{u_1,u,x_3\}$ and $Z = \{u,v\}$, there is a 5-colouring ϕ of $G \setminus \{u,v\}$ such that $\phi(u_1) = \phi(x_3)$ and $\phi(y) = \phi(x_1)$ for all $y \in A$. Choose $\alpha_1 \in \{1,\ldots,5\}$ with $\alpha_1 \neq \phi(x_1),\phi(x_3)$, and $\phi(y)$ for all $y \in N - A$; and choose $\alpha_2 \in \{1,\ldots,5\}$ with $\alpha_2 \neq \alpha_1,\phi(x_1),\phi(x_3)$, $\phi(u_2)$. Setting $\phi(u) = \alpha_2$ and $\phi(v) = \alpha_1$ defines a 5-colouring of G, a contradiction. This proves (1).

Let $N = \{v_1,v_2,v_3\}$. Since x_2x_3 are adjacent, and vx_2 is in ≤ 3 triangles, it follows that at most one of v_1,v_2,v_3 is adjacent to x_2; and since vx_1 is in ≤ 3 triangles, at most two of v_1,v_2,v_3 are adjacent to x_1. By (1) we may therefore assume that v_1x_1,v_2x_1,v_3x_2 are adjacent, and hence v_1x_2,v_2x_2,v_3x_1 are non-adjacent. Since vx_1 is in ≤ 3 triangles, x_1x_3 are non-adjacent. By (1) with x_2 and x_3 exchanged, v_3x_3 are adjacent, and v_1x_3,v_2x_3 are therefore non-adjacent since vx_3 is in ≤ 3 triangles.

Since $\{u,v,x_2,x_3\}$ and $\{v,x_2,x_3,v_3\}$ are 4-cliques, it follows that $\{v,x_1,v_1,v_2\}$ is not a 4-clique, because v is in ≤ 2 4-cliques by (3.7). Hence v_1v_2 are not adjacent, and so $\{v_1,v_2,x_2\}$ is stable. By (5.1) with $X_1 = \{v_1,v_2,x_2,v\}$ and $X_2 = \{x_1,x_3,u\}$, there is a 5-colouring ϕ of $G \setminus \{u,v\}$ such that $\phi(v_1) = \phi(v_2) = \phi(x_2)$ and $\phi(x_1) = \phi(x_3)$. Choose $\alpha_1 \in \{1,\ldots,5\}$ with $\alpha_1 \neq \phi(x_1),\phi(x_2),\phi(u_1),\phi(u_2)$, and choose $\alpha_2 \in \{1,\ldots,5\}$ with $\alpha_2 \neq \alpha_1,\phi(x_1),\phi(x_2),\phi(v_3)$; then setting $\phi(u) = \alpha_1$ and $\phi(v) = \alpha_2$ defines a 5-colouring of G, a contradiction. ∎

(5.5) *Let G be a non-apex Hadwiger graph; then every 4-clique of G contains at most one 6-valent vertex.*

Proof. Let $\{u,v,x_1,x_2\}$ be a 4-clique, and suppose that u,v are both 6-valent. By (5.4) uv is in exactly 2 triangles. Let the neighbours of u be v,x_1,x_2,u_1,u_2,u_3, and let the neighbours of v be u,x_1,x_2,v_1,v_2,v_3 where $u_1,u_2,u_3 \neq v_1,v_2,v_3$. By (5.2), $\{u_1,u_2,v\}$ is not stable, and so u_1u_2 are adjacent, and similarly u_1u_3 and u_2u_3 are adjacent. Hence $\{u,u_1,u_2,u_3\}$ is a 4-clique, and similarly so is $\{v,v_1,v_2,v_3\}$, and so is $\{u,v,x_1,x_2\}$, contrary to (4.4). ∎

We deduce

(5.6) *Let G be a non-apex Hadwiger graph. Then at most two vertices of G have valency 6, and all others have valency ≥ 7.*

Proof. By (5.3) every 6-valent vertex belongs to two 4-cliques, and by (5.5) every 4-clique contains at most one 6-valent vertex. From (4.5) there are at most four 4-cliques, and the result follows. ∎

This concludes step 1 of the proof sketched in the introduction.

6. Separations of order 6

The second step in the main proof is to show that every non-apex Hadwiger graph is 7-connected except for its (≤ 2) 6-valent vertices. In this section, we begin to investigate possible 6-separations. First, we need a trivial strengthening of a result of Mader [7].

(6.1) *If G is a simple graph with $|V(G)| \geq 4$ and with no K_6-minor, then $|E(G)| \leq 4|V(G)| - 10$. Moreover, if equality holds and $|V(G)| \geq 5$ then every edge of G is in ≥ 3 triangles.*

Proof. The inequality was proved by Mader [7]. Suppose that equality holds and $|V(G)| \geq 5$, and let $e \in E(G)$ be in T triangles. Form H from G by contracting e and deleting the T parallel edges that result; then

$$|E(H)| = |E(G)| - T - 1 = (4|V(G)| - 10) - T - 1.$$

From Mader's inequality applied to H,

$$|E(H)| \leq 4|V(H)| - 10 = 4|V(G)| - 14,$$

and so $T \geq 3$, as required. ∎

The following result is mainly for reassurance.

(6.2) *Every non-apex Hadwiger graph has ≥ 18 vertices.*

Proof. Let G be a non-apex Hadwiger graph with n vertices. By (5.6), $2|E(G)| \geq 7n - 2$; but by (6.1), $|E(G)| \leq 4n - 10$. Hence $2(4n - 10) \geq 7n - 2$, and so $n \geq 18$. ∎

(6.3) *Let (A,B) be a 6-separation of a non-apex Hadwiger graph G, with $|A - B| \geq 2$ and $|B - A| \geq 1$. Then $|A - B| \geq 5$.*

Proof. Suppose first that $|A-B| = 2, A-B = \{a_1,a_2\}$ say. Since a_1,a_2 have valency ≥ 6 there are ≥ 4 vertices in $A \cap B$ adjacent to both a_1 and a_2. By (2.7), a_1a_2 are non-adjacent; and so a_1 and a_2 are 6-valent and both are adjacent to every vertex in $A \cap B$. By (5.3), $G|A \cap B$ has a triangle, with vertex set $\{v_1,v_2,v_3\}$ say. Let C

be a component of $G \mid (B - A)$. Since G is 6-connected, every vertex in $A \cap B$ has a neighbour in C. Let $v_4, v_5 \in A \cap B - \{v_1, v_2, v_3\}$ be distinct. Then

$$\{\{v_1\}, \{v_2\}, \{v_3\}, \{a_1, v_4\}, \{a_2, v_5\}, V(C)\}$$

is a 6-cluster in G, a contradiction.

Consequently, $|A - B| \geq 3$, and so by (5.6) there is a vertex $a_1 \in A - B$ with valency ≥ 7. It therefore has a neighbour a_2 in $A - B$. Since $A - \{a_1, a_2\}$ contains every neighbour of a_1 except a_2, and every neighbour of a_2 except a_1, it follows that the edge a_1a_2 is in at least $\delta(a_1) + \delta(a_2) - |A|$ triangles, where $\delta(a_i)$ denotes the valency of a_i. By (2.7), $3 + |A| \geq \delta(a_1) + \delta(a_2)$, and if equality holds, then by (5.4), $\delta(a_1) + \delta(a_2) \geq 14$. Since in any case, $\delta(a_1) + \delta(a_2) \geq 13$, we deduce that $3 + |A| \geq 14$, that is, $|A - B| \geq 5$, as required. ∎

If (A, B) is a separation of G, we define $\eta(A, B)$ to be the maximum h such that there exist $|A \cap B|$ disjoint fragments X_i $(1 \leq i \leq |A \cap B|)$ of $G \mid B$, each containing one vertex of $A \cap B$, and there are h pairs i, j with $1 \leq i < j \leq |A \cap B|$ and X_iX_j adjacent.

(6.4) *Let (A, B) be a separation of a simple graph G of order $k \geq 4$, let $v \in B - A$, and let there be k paths of $G \mid B$ between v and $A \cap B$, mutually disjoint except for v. Suppose that there is no separation (C, D) of $G \mid B$ with $C \cap D = \{v\}$ and $|C \cap A|, |D \cap A| \geq 2$. Then $\eta(A, B) \geq 2k - 3$; and if there is a circuit in $G \mid A \cap B$ of length $< k$, then $\eta(A, B) \geq 2k - 2$.*

Proof. Let $\mathcal{P}$ be a set of k paths of $G \mid B$ from v to $A \cap B$, mutually disjoint except for v. Since $k \geq 4$, we may partition $\mathcal{P}$ into two sets $\mathcal{P}_1, \mathcal{P}_2$ both of cardinality ≥ 2; and from the non-existence of (C, D) as in the theorem, there is a path of $G \mid (B - \{v\})$ from some member of $\mathcal{P}_1$ to some member of $\mathcal{P}_2$. Consequently, there exists $P_1 \in \mathcal{P}$ such that there is a path of $G \mid (B - \{v\})$ from $V(P_1)$ to

$$\bigcup (V(P) : P \in \mathcal{P} - \{P_1\}) - \{v\}.$$

Define $X_1 = V(P_1) - \{v\}$. We define $X_2, \ldots, X_{k-1}$ and $P_2, \ldots, P_{k-1} \in \mathcal{P}$ inductively, as follows. Suppose that $2 \leq j \leq k - 1$, and we have defined fragments $X_1, \ldots, X_{j-1}$ and paths $P_1, \ldots, P_{j-1} \in \mathcal{P}$, in such a way that $X_1, \ldots, X_{j-1} \subseteq B - \{v\}$ and are mutually disjoint, and $V(P_i) - \{v\} \subseteq X_i$ for $1 \leq i \leq j - 1$, and for $2 \leq i \leq j - 1$ some vertex of X_i is adjacent to a vertex in $X_1 \cup \ldots \cup X_{i-1}$. We shall define X_j, P_j using (1).

(1) *There is a path Q of $G \mid (B - \{v\})$ from*

$$\bigcup (V(P) : P \in \mathcal{P} - \{P_1, \ldots, P_{j-1}\}) - \{v\}$$

to $X_1 \cup \ldots \cup X_{j-1}$.

For if not, then $j \geq 3$ (from our choice of P_1), and there is a separation (C, D) of $G \mid B$ with $C \cap D = \{v\}$, $X_1 \cup \ldots \cup X_{j-1} \subseteq C$, and $V(P) \subseteq D$ for all $P \in \mathcal{P} - \{P_1, \ldots, P_{j-1}\}$. Since $j \geq 3$ it follows that $|C \cap A| \geq j - 1 \geq 2$; and since $j < k$ it follows that

$$|D \cap A| \geq |\mathcal{P} - \{P_1, \ldots, P_{j-1}\}| = k - (j - 1) \geq 2.$$

But then (C,D) contradicts the hypothesis. This proves (1).

To complete the definition of X_j and P_j, choose Q as in (1) with Q minimal, with ends a,b, where $a \in V(P_j)$ for some $P_j \in \mathcal{P} - \{P_1,\dots,P_{j-1}\}$ and $b \in X_1 \cup \dots \cup X_{j-1}$. Define $X_j = V(P_j \cup Q) - \{b,v\}$. This completes the definition of X_j and P_j. We see that X_j is disjoint from $X_1,\dots,X_{j-1}$, that X_j is a fragment, that $V(P_j) - \{v\} \subseteq X_j$, and that some vertex in X_j is adjacent to a vertex in $X_1 \cup \dots \cup X_{j-1}$.

Let $\{P_k\} = \mathcal{P} - \{P_1,\dots,P_{k-1}\}$, and let $X_k = V(P_k)$. Since $v \in X_k$ it follows that X_iX_k are adjacent for $1 \le i \le k-1$; and for $2 \le j \le k-1$ there exists i with $1 \le i < j$ such that X_iX_j are adjacent. Consequently, there are $\ge 2k-3$ adjacent pairs altogether, and so $\eta(A,B) \ge 2k-3$. This proves the first claim of the theorem.

For the second, suppose that $v_1,\dots,v_h \in A \cap B$ are the vertices of a circuit in order, where $h < k$. Choose $\mathcal{P}$ as before, and for $1 \le i \le h$ let $P_i \in \mathcal{P}$ have ends v, v_i. Then setting $X_i = V(P_i) - \{v\}$ for $1 \le i \le h$ satisfies the conditions of the inductive definition, and so we may choose $X_{h+1},\dots,X_k$ as before. Then, as before, there are $\ge 2k-3$ pairs i,j with $1 \le i < j \le k$ such that X_iX_j are adjacent, counting only one pair i,j for each value of $j < k$. But for $j = h$, there are two pairs i,j namely $1,j$ and $j-1,j$; and so in total there are $\ge 2k-2$ pairs. ∎

(6.5) *Let (A,B) be a k-separation with $k \ge 6$ of a non-apex Hadwiger graph G, and let $Z \subseteq A \cap B$ with $|Z| = z \ge 2$. Define $\delta = 0$ if some vertex in $A - B$ has valency 6, and $\delta = 1$ otherwise. Define $\varepsilon = 0$ if every vertex in Z has ≤ 2 neighbours in $A - B$ and there are $\le z$ vertices in $A - B$ with a neighbour in Z, and $\varepsilon = 1$ otherwise. Then either*

(i) *some vertex in Z has at most one neighbour in $A - B$, or*

(ii) $\eta(A,B) + z + \delta + \varepsilon \le 4k - 12$, *or*

(iii) *there are two 6-valent vertices in $A - B$ both with no neighbour in $A - B$, or*

(iv) *let H be the subgraph of G with $V(H) = (A - B) \cup Z$ and with $E(H) = E(G \mid V(H)) - E(G \mid Z)$; then H cannot be drawn in a plane so that every vertex in Z is incident with the infinite region.*

Proof. We assume that (i), (iii) and (iv) are false. Let $|A - B| = n$. Let there be α edges of G with both ends in $A - B$, β edges with one end in $A - B$ and the other in Z, and γ with one end in $A - B$ and the other in $A \cap B - Z$. Define $\varepsilon' = 0$ if some edge of G with both ends in $A - B$ is in ≤ 2 triangles, and $\varepsilon' = 1$ otherwise.

(1) $2\alpha + \beta + \gamma \ge 7n + \delta + \varepsilon' - 2$.

For suppose the inequality is false. Then, if $\delta(v)$ denotes the valency of a vertex v, we have

$$\sum(\delta(v) \quad v \in A - B) = 2\alpha + \beta + \gamma \le 7n + \delta + \varepsilon' - 3.$$

Hence some vertex in $A - B$ has valency 6, and so $\delta = 0$. Therefore, from the same inequality, at least two vertices a_1, a_2 in $A - B$ have valency 6, and by (5.6) all other vertices in $A - B$ are 7-valent. Since (iii) is false, we may assume that a_1 has a neighbour $a_3 \in A - B$. Then a_3 is 6- or 7-valent, and so by (5.4), a_1a_3 is in ≤ 2 triangles; and hence $\varepsilon' = 0$, and the inequality of (1) holds.

(2) $\alpha - \eta(A,B) \geq 3n - 4k + \delta + 9$.

For let $X_1, \ldots, X_k$ be disjoint fragments of $G \mid B$ each containing one vertex of $A \cap B$, such that $X_i X_j$ are adjacent for $\eta(A,B)$ pairs i,j with $1 \leq i < j \leq k$. Let J be obtained from G by deleting all vertices in $B - X_1 \cup \ldots \cup X_k$, contracting all edges with both ends in X_i for $1 \leq i \leq k$, and deleting any parallel edges. Then J is simple, and has $n+k$ vertices and $\alpha + \beta + \gamma + \eta(A,B)$ edges. Since $k \geq 6$, it follows from (6.1) that

$$\alpha + \beta + \gamma + \eta(A,B) \leq 4(n+k) - 10$$

with equality only if every edge of J is in ≥ 3 triangles. In particular if $\varepsilon' = 0$ then equality does not hold, and so

$$\alpha + \beta + \gamma + \eta(A,B) \leq 4(n+k) - 11 + \varepsilon'.$$

Then (2) follows from (1) by subtracting.

(3) $\alpha \leq 3n - z - 3 - \varepsilon$.

For let H be as in (iv); since (i) and (iv) are false, H can be drawn in the plane so that every vertex in Z is incident with the infinite region, and every vertex in Z has valency ≥ 2 in H. Let there be $z + z'$ vertices incident with the infinite region in the drawing of H. Since Z is stable in H, we may add $2z - 3$ new edges to H joining pairs of vertices in Z so that the result, H' say, is still simple and planar. Consequently,

$$|E(H)| + 2z - 3 = |E(H')| \leq 3(n+z) - 6,$$

and so $|E(H)| \leq 3n + z - 3$. Also, since every vertex in Z has ≥ 2 neighbours in $A - B$, it follows that $\beta \geq 2z$. Suppose that we have equality in both; that is, $|E(H)| = 3n + z - 3$ and $\beta = 2z$. It follows that H' is a planar triangulation, and so $z' \leq z$; but also, every vertex in $A - B$ with a neighbour in Z is incident with the infinite region since every vertex in Z is 2-valent (because $\beta = 2z$), and so there are $\leq z' \leq z$ such vertices. Hence if we have equality in both inequalities then $\varepsilon = 0$; and so,

$$(3n + z - 3 - |E(H)|) + (\beta - 2z) \geq \varepsilon.$$

Since $|E(H)| = \alpha + \beta$ this proves (3).

By combining (2) and (3), we deduce that (ii) holds. ∎

(6.6) *Let G be a non-apex Hadwiger graph, and let (A,B) be a 6-separation with $|A - B| \geq 2$ and $|B - A| \geq 2$. Let $A \cap B = \{v_1, \ldots, v_6\}$, and let H be the subgraph of G with $V(H) = A - \{v_5, v_6\}$ and $E(H) = E(G \mid V(H)) - E(G \mid \{v_1, \ldots, v_4\})$. Then there is a cluster in H traversing $\{v_1, v_2, v_3, v_4\}$.*

Proof. We proceed by induction on $|A|$.

(1) *We may assume that $v_1, \ldots, v_4$ all have valency ≥ 2 in H.*

For suppose that for some $v \in A - B$, v_1 has no neighbour in $A - (B \cup \{v\})$. Then $(A - \{v_1\}, B \cup \{v\})$ is a 6-separation of G, and $|(A - \{v_1\}) - (B \cup \{v\})| \geq 2$ since $|A - B| \geq 3$ by (6.3). From the inductive hypothesis there is a 4-cluster $\{X_1, X_2, X_3, X_4\}$ of $H \backslash v_1$ with $v \in X_1$ and $v_i \in X_i$ $(i = 2,3,4)$. But then $\{X_1 \cup \{v_1\}, X_2, X_3, X_4\}$ satisfies the theorem. This proves (1).

(2) *We may assume that there is no trisection* (C_1, C_2, D) *of* H *of order* 2 *with*

$$|(C_1 - D) \cap \{v_1, \ldots, v_4\}| = |(C_2 - D) \cap \{v_1, \ldots, v_4\}| = 1.$$

For suppose that C_1, C_2, D is such a trisection, with $v_1 \in C_1 - D, v_2 \in C_2 - D$, and $v_3, v_4 \in D$ say. Let $C_1 \cap C_2 \cap D = \{a_1, a_2\}$. Since $(C_1, C_2 \cup D)$ is a 2-separation of H, it follows that $(C_1 \cup \{v_5, v_6\}, C_2 \cup D \cup B)$ is a 5-separation of G. Consequently, $C_2 \cup D \cup B = V(G)$, and similarly $C_1 \cup D \cup B = V(G)$. Hence $D = (A - B) \cup \{v_3, v_4\}$. Since $v_1 \in C_1 - D$ and v_1 has two neighbours in $A - B$ which therefore belong to C_1, it follows that $a_1, a_2 \in A - B$, and a_1, a_2 are both adjacent to v_1; and similarly they are both adjacent to v_2. Now $(B \cup \{a_1, a_2\}, A - \{v_1, v_2\})$ is a 6-separation of G, and

$$|(A - \{v_1, v_2\}) - (B \cup \{a_1, a_2\})| \geq 2$$

since $|A - B| \geq 4$ by (6.3). From the inductive hypothesis, there is a 4-cluster $\{X_1, X_2, X_3, X_4\}$ of $H \setminus \{v_1, v_2\}$ with $a_1 \in X_1, a_2 \in X_2$, $v_3 \in X_3, v_4 \in X_4$; but then

$$\{X_1 \cup \{v_1\}, X_2 \cup \{v_2\}, X_3, X_4\}$$

satisfies the theorem. This proves (2).

(3) *There is no* (≤ 3)*-separation* (C, D) *of* H *with* $v_1, \ldots, v_4 \in C$, $|D - C| \geq 2$, *and* $|\{v_1, \ldots, v_4\} \cap D| \leq 2$.

For if (C, D) is such a separation, then $(C \cup B, D \cup \{v_5, v_6\})$ is a (≤ 5)-separation of G, and yet $B \cup C \neq V(G)$ since $|D - C| \geq 1$, a contradiction. This proves (3).

(4) $\eta(A, B) \geq 9$.

Let us apply (6.4), taking $k = 6$. Choose $v \in B - A$ arbitrarily; then by the 6-connectivity of G, the k paths of (6.4) exist. We claim there is no separation (C, D) of $G \mid B$ with $C \cap D = \{v\}$ and $|C \cap A|, |D \cap A| \geq 2$. For suppose that (C, D) is such a separation. Then $(C \cup A, D)$ is a separation of G, of order

$$|C \cap D| + |A \cap D| = |C \cap D| + 6 - |A \cap C| \leq 5$$

and so $C \cup A = V(G)$; and similarly $D \cup A = V(G)$. Hence $B - A = \{v\}$, a contradiction. Thus there is no such (C, D), and the claim follows from (6.4).

(5) *H cannot be drawn in a disc with* v_1, v_2, v_3, v_4 *on the boundary in some order.*

For let us apply (6.5), taking $k = 6$ and $Z = \{v_1, v_2, v_3, v_4\}$. Certainly (6.5)(i) is false, by (1), and (6.5)(ii) is false, by (4). Also, (6.5)(iii) is false, for otherwise there would be a 6-separation (C, D) of G with $|C - D| = 2$ and $|D - C| \geq 2$, contrary to (6.3). Thus (6.5)(iv) holds, as required.

From (2.6) (applied to H), (2), (3), and (5), we deduce the theorem. ∎

(6.7) *Let* (A, B) *be a 6-separation of a non-apex Hadwiger graph* G, *with* $|A - B|$, $|B - A| \geq 2$. *Then* $G \mid A \cap B$ *has no circuit of length* 4.

Proof. Suppose that $A \cap B = \{v_1, \ldots, v_6\}$, where $v_1v_2, v_2v_3, v_3v_4, v_4v_1$ are adjacent. By (6.6), there is a cluster $\{X_1, X_3, X_5, X_6\}$ of $G \mid (A - \{v_2, v_4\})$ with $v_i \in X_i$ $(i =$

1,3,5,6); and there is a cluster $\{Y_2, Y_4, Y_5, Y_6\}$ of $G \mid (B - \{v_1, v_3\})$ with $v_i \in Y_i$ ($i =$ 2,4,5,6). But then

$$\{X_1, Y_2, X_3, Y_4, X_5 \cup Y_5, X_6 \cup Y_6\}$$

is a 6-cluster in G, a contradiction. ∎

(6.8) *Let G be a non-apex Hadwiger graph, and let $W \subseteq V(G)$ with $|W| = 6$. Then $G \setminus W$ has ≤ 2 components.*

Proof. Let the vertex sets of the components of $G \setminus W$ be $C_1, \ldots, C_k$, and suppose that $k \geq 3$. Now $|C_i| = 1$ for at most one value of i, since if $|C_1| = |C_2| = 1$ say then the separation

$$(C_1 \cup C_2 \cup W, C_3 \cup \ldots \cup C_k \cup W)$$

fails to satisfy (6.3). In particular we may assume that $|C_1| > 1$. By (6.6) there is a 4-cluster $\{X_1, X_2, X_3, X_4\}$ in $G \mid (W \cup C_1)$ with $v_i \in X_i$ ($1 \leq i \leq 4$), where $W = \{v_1, \ldots, v_6\}$. Then

$$\{X_1, X_2, X_3, X_4, C_2, C_3 \cup \{v_5\}\}$$

is a 6-cluster, a contradiction. ∎

Let G be a graph, let $Z \subseteq V(G)$ with $|Z| = 6$, and let $v_1, v_2, v_3 \in Z$ be distinct. An *octopus on Z in G with base* v_1, v_2, v_3 is a set of eight disjoint fragments of G, that can be numbered $\{X_1, \ldots, X_8\}$ so that

(i) $v_i \in X_i$ ($1 \leq i \leq 3$) and $|Z \cap X_i| = 1$ ($4 \leq i \leq 6$)

(ii) for $1 \leq i \leq 3, X_i X_7$ and $X_i X_8$ are both adjacent

(iii) for $4 \leq i \leq 6$, one of $X_i X_7, X_i X_8$ is adjacent

(iv) $X_7 X_8$ are adjacent.

(See figure 2, where each X_i has been contracted to a single vertex. This shows one of the two basic types of octopus; in the other type, X_4, X_5, X_6 are all adjacent to X_7 and not to X_8, or vice versa.)

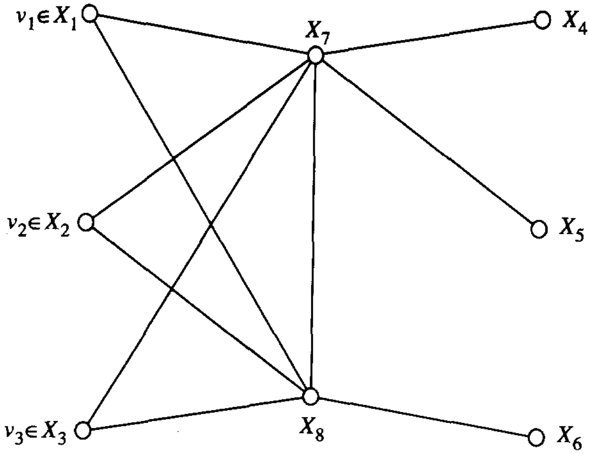

Fig. 2. An octopus

(6.9) *Let (A,B) be a 6-separation of a non-apex Hadwiger graph G, with $|A-B| \geq 2$ and $|B-A| \geq 2$. Let $A \cap B = \{v_1,\dots,v_6\}$. Then there is an octopus in $G \mid A$ on $A \cap B$ with base v_1, v_2, v_3.*

Proof. We proceed by induction on $|A|$.

(1) *We may assume that there is no 6-separation (A',B') of G with $A' \subseteq A$ and $B \subseteq B'$ such that $|A'-B'| \geq 2$ and $|A'| < |A|$.*

For if (A',B') is such a separation, by Menger's theorem, there are six disjoint paths $P_1,\dots,P_6$ of $G \mid (A \cap B')$, where P_i has ends v_i and $v_i' \in A' \cap B'$ say, for $1 \leq i \leq 6$. From the inductive hypothesis, there is an octopus $\{X_1',\dots,X_8'\}$ in $G \mid A'$ on $A' \cap B'$ with base v_1', v_2', v_3', where $v_i' \in X_i'$ $(1 \leq i \leq 6)$. Let $X_i = X_i' \cup V(P_i)$ $(1 \leq i \leq 6)$ and $X_i = X_i'$ $(i=7,8)$; then $\{X_1,\dots,X_8\}$ satisfies the theorem.

From (1) and (6.3) it follows that

(2) *v_1, v_2, v_3 all have ≥ 2 neighbours in $A-B$.*

Moreover,

(3) *There is no (≤ 3)-separation (C,D) of $G \mid (A - \{v_4,v_5,v_6\})$ with $v_1,v_2,v_3 \in C$ and $|D-C| \geq 2$ and $D \neq A - \{v_4,v_5,v_6\}$.*

For if (C,D) is such a separation, $(C \cup B, D \cup \{v_4,v_5,v_6\})$ is a separation of G of order ≤ 6. By (1), $D \cup \{v_4,v_5,v_6\} = A$, a contradiction.

(4) *There are ≥ 4 vertices in $A-B$ with a neighbour in $\{v_1,v_2,v_3\}$.*

For let the set of such vertices be N. Then $(A - \{v_1,v_2,v_3\}, B \cup N)$ is a separation of G, of order $|N|+3$. Suppose that $|N| \leq 3$. Then this separation has order ≤ 6, and yet

$$|(A - \{v_1,v_2,v_3\}) - (B \cup N)| \geq 2$$

since $|A-B| \geq 5$ by (6.3). This contradicts (1).

(5) *$G \mid (A - \{v_4,v_5,v_6\})$ cannot be drawn in a disc with v_1,v_2,v_3 on the boundary.*

For let us apply (6.5), taking $k=6$ and $Z = \{v_1,v_2,v_3\}$, and $\varepsilon = 1$ (by (4)). (6.5)(i) does not hold, by (2); (6.5)(ii) does not hold, since $\eta(A,B) \geq 9$ by (6.4); and (6.5)(iii) does not hold, by (6.8). Thus (6.5)(iv) holds, and (5) follows.

From (5), (3), (3.4), (3.5) and the 6-connectivity of G, there is a legless tripod in $G \mid (A - \{v_4,v_5,v_6\})$ with feet v_1,v_2,v_3. Consequently, there are disjoint fragments $X,Y \subseteq A-B$ of G such that X and Y both contain neighbours of v_1,v_2 and v_3. Choose X and Y with $X \cup Y$ maximal; then every vertex in $V(G) - X \cup Y$ with a neighbour in $X \cup Y$ belongs to $A \cap B$, from the maximality of $X \cup Y$, and hence v_4,v_5,v_6 all have a neighbour in $X \cup Y$. Moreover, by (6.8), XY are adjacent. Consequently,

$$\{\{v_1\},\{v_2\},\{v_3\},\{v_4\},\{v_5\},\{v_6\},X,Y\}$$

is the desired octopus. ∎

7. Reductions for 6-separations

Now we use the results of the last section to eliminate most possibilities for 6-separations. We begin with the following lemma.

(7.1) *Let G be a graph, let $Z \subseteq V(G)$ with $|Z| = 5$, and suppose that X_1, X_2 is feasible in G for all disjoint $X_1, X_2 \subseteq Z$ with $|X_1| = |X_2| = 2$. Then there is at most one $X \subseteq Z$ with $|X| = 2$ such that $X, Z - X$ is infeasible in G.*

Proof. Let $Z = \{z_1, \ldots, z_5\}$ and suppose that $\{z_1, z_2\}, \{z_3, z_4, z_5\}$ is infeasible.

(1) *$X, Z - X$ is feasible for all $X \subseteq \{z_3, z_4, z_5\}$ with $|X| = 2$.*

For let $X = \{z_3, z_4\}$ say. Since $\{z_1, z_2\}, \{z_3, z_4\}$ is feasible, there are disjoint connected subgraphs H_1, H_2 with $z_1, z_2 \in V(H_1)$ and $z_3, z_4 \in V(H_2)$. Since $\{z_1, z_2\}, \{z_4, z_5\}$ is feasible, there is a path from z_5 to z_4. Hence there is a minimal path Q from z_5 to $V(H_1 \cup H_2)$. If the end of Q is in $V(H_1)$ then $\{z_1, z_2, z_5\}, \{z_3, z_4\}$ is feasible as required; and if the end of Q is in $V(H_2)$ then $\{z_1, z_2\}, \{z_3, z_4, z_5\}$ is feasible, a contradiction. This proves (1).

In view of (1), we may suppose for a contradiction that $\{z_1, z_3\}, \{z_2, z_4, z_5\}$ is infeasible. Hence there is symmetry between z_2 and z_3.

(2) *There is a path P between z_2 and z_3, and a path Q between z_4 and z_5, with $V(P \cap Q) = \emptyset$, and a path R from z_1 to an internal vertex of P, with $|V(R \cap P)| = 1$ and $V(R \cap Q) = \emptyset$.*

For since $\{z_1, z_2\}, \{z_4, z_5\}$ is feasible, there are disjoint paths S, Q with ends z_1, z_2 and z_4, z_5 respectively. Since $\{z_3, z_4\}, \{z_1, z_2\}$ is feasible, there is a path from z_3 to $V(Q)$ in $G \backslash \{z_1, z_2\}$. Hence there is a path from z_3 to $V(Q \cup S)$ in $G \backslash \{z_1, z_2\}$. Take a minimal such path T, and let its ends be z_3, r. Now $r \notin V(Q)$ since $\{z_1, z_2\}$, $\{z_3, z_4, z_5\}$ is infeasible. Consequently, $r \in V(S)$. Let P be the path in $S \cup T$ between z_2 and z_3, and let R be the subpath of S from z_1 to r; then (2) holds.

Choose P, Q, R as in (2) with $|E(R)|$ minimum. Let R have ends z_1, r. Now since $\{z_2, z_4\}, \{z_3, z_5\}$ is feasible, there are two disjoint paths from $V(P)$ to $V(Q)$, and hence there is one, S say, with $r \notin V(S)$. Choose such a path S, minimal, with ends $p \in V(P)$ and $q \in V(Q)$. Since $r \notin V(S)$, we may assume from the symmetry that p lies in the component of $P \backslash r$ containing z_2. Then S has no vertex in P except p, and none in Q except q, by the minimality of S.

Now $S \cap R$ is null; for otherwise, let s be the vertex of $S \cap R$ closest to p in S, and let P' be the union of the subpath of P from z_2 to p, the subpath of S from p to s, the subpath of R from s to r, and the subpath of P from r to z_3; and let R' be the subpath of R from z_1 to s. Then P', Q, R' satisfy (2), contrary to the minimality of $|E(R)|$. This proves that R and S are disjoint.

Let H_1 be the union of R and the subpath of P from r to z_3, and let H_2 be the union of Q, S and the subpath of P from z_2 to p. Then H_1, H_2 are disjoint and connected, and so $\{z_1, z_3\}, \{z_2, z_4, z_5\}$ is feasible, a contradiction. The result follows. ∎

(7.1) is best possible in the sense that there may be one X as in (7.1) with $X, Z - X$ infeasible. For example, let G' be a graph which can be drawn in the plane, and let $z_1, z_2, z_3, z_4, z_5, z_6$ be vertices incident with the infinite region, in order. Let

z_6 be 4-valent, with neighbours a,b,c,d in order. Let G be obtained from G' by deleting z_6 and adding edges ac and bd. Then if G is sufficiently connected, it satisfies the hypotheses of (7.1) with $Z=\{z_1,\dots,z_5\}$, and yet $\{z_1,z_3,z_5\},\{z_2,z_4\}$ is infeasible. The existence of this construction will give us a lot of trouble.

Throughout the remainder of this section, G is a non-apex Hadwiger graph, and (A,B) is a 6-separation of G with $|A-B|,|B-A|\geq 2$. Let $A\cap B=\{v_1,\dots,v_6\}$. From (6.6), we have

(7.2) *For all disjoint $X_1,X_2\subseteq A\cap B$ with $|X_1|=|X_2|=2$, X_1,X_2 is feasible in $G\,|\,((A-B)\cup X_1\cup X_2)$ and in $G\,|\,((B-A)\cup X_1\cup X_2)$.*

Consequently, from (7.1) we have

(7.3) *For all $Z\subseteq A\cap B$ with $|Z|=5$, there is at most one $X\subseteq Z$ with $|X|=2$ such that $X,Z-X$ is infeasible in $G\,|\,((B-A)\cup Z)$, and at most one such that $X,Z-X$ is infeasible in $G\,|\,((A-B)\cup Z)$.*

On the other hand, we have

(7.4) *Let $Z_1,\dots,Z_k$ be a partition of $A\cap B$ into stable sets, such that Z_iZ_j are adjacent for $1\leq i<j\leq k$. Then $Z_1,\dots,Z_k$ is infeasible in one of $G\,|\,A,G\,|\,B$.*

Proof. Suppose that $Z_1,\dots,Z_k$ is feasible in $G\,|\,A$, via $X_1,\dots,X_k$. By (5.1) there is a 5-colouring ϕ_2 of $G\,|\,B$ such that for $1\leq i\leq k, \phi_2(u)=\phi_2(v)$ for all $u,v\in Z_i$, and for $1\leq i<j\leq k$, $\phi_2(u)\neq\phi_2(v)$ for $u\in Z_i$ and $v\in Z_j$. Hence we may assume that $\phi_2(u)=i$ for $u\in Z_i$ $(1\leq i\leq k)$. Similarly if $Z_1,\dots,Z_k$ is feasible in $G\,|\,B$, there is an analogous 5-colouring ϕ_1 of $G\,|\,A$. Let $\phi(v)=\phi_1(v)$ if $v\in A$, and $\phi(v)=\phi_2(v)$ if $v\in B$; then ϕ is a 5-colouring of G, a contradiction. ∎

(7.5) *$A\cap B$ is not the union of a clique and a stable set.*

Proof. Suppose that $A\cap B=X\cup Y$, where $X\cap Y=\emptyset$, $G\,|\,X$ is complete, and Y is stable. Choose Y maximal; then each $v\in X$ has a neighbour in Y. But the partition of $A\cap B$ into Y and the sets $\{v\}(v\in X)$ is feasible in both $G\,|\,A$ and $G\,|\,B$, contrary to (7.4). ∎

The following is a generalization of (7.4).

(7.6) *Let $Z_1,\dots,Z_k$ be a partition of $A\cap B$ into stable sets, where $k\geq 3$ and Z_iZ_j are adjacent for all i, j with $1\leq i<j\leq k$ except possibly for $(i,j)=(1,2)$, $(1,3)$. Then either*

(i) *there do not exist disjoint fragments $X_1,\dots,X_k$ of $G\,|\,A$ with $Z_i\subseteq X_i$ $(1\leq i\leq k)$ such that X_1X_2 are adjacent, or*

(ii) *there do not exist disjoint fragments $Y_1,\dots,Y_k$ of $G\,|\,B$ with $Z_i\subseteq Y_i$ $(1\leq i\leq k)$ such that Y_1Y_3 are adjacent.*

Proof. Suppose $X_1,\dots,X_k$ exist as in (i). By (5.1) there is a 5-colouring ϕ_2 of $G\,|\,B$ such that for $1\leq i\leq k,\phi_2(u)=\phi_2(v)$ for all $u,v\in Z_i$; and moreover, if $\phi_2(Z_i)$ denotes the common value of $\phi_2(u)$ for $u\in Z_i$, then $\phi_2(Z_i)\neq\phi_2(Z_j)$ for all i,j with $1\leq i<j\leq k$ except possibly $(i,j)=(1,3)$. Now suppose also that $Y_1,\dots,Y_k$ exist as in (ii); then similarly there is a 5-colouring ϕ_1 of $G\,|\,A$ and values $\phi_1(Z_i)$ $(1\leq i\leq k)$ such that for $1\leq i\leq k,\phi_1(u)=\phi_1(Z_i)$ for all $u\in Z_i$ and for $1\leq i<j\leq k$, $\phi_1(Z_i)\neq\phi_1(Z_j)$ except possibly for $(i,j)=(1,2)$.

If $\phi_1(Z_1) \neq \phi_1(Z_2)$ and $\phi_2(Z_1) \neq \phi_2(Z_3)$, we may assume that $\phi_1(Z_i) = \phi_2(Z_i) = i$ for $1 \leq i \leq k$; but then setting $\phi(v) = \phi_1(v)$ $(v \in A)$ and $\phi(v) = \phi_2(v)$ $(v \in B)$ defines a 5-colouring of G, a contradiction. We may therefore assume that $\phi_1(Z_1) = \phi_1(Z_2)$, and hence $Z_1 \cup Z_2$ is stable. Now $Z_1 \cup Z_2, Z_3, \ldots, Z_k$ is feasible in $G \mid A$, via $X_1 \cup X_2, X_3, \ldots, X_k$, since X_1X_2 are adjacent. By (5.1) there is a 5-colouring ϕ_3 of $G \mid B$ and values $\phi_3(Z_1 \cup Z_2), \phi_3(Z_3), \ldots, \phi_3(Z_k)$ such that $\phi_3(u) = \phi_3(Z_1 \cup Z_2)$ for all $u \in Z_1 \cup Z_2$, and $\phi_3(u) = \phi_3(Z_i)$ for all $u \in Z_i$ $(3 \leq i \leq k)$. Since $Z_1 \cup Z_2, Z_3, \ldots, Z_k$ are mutually adjacent, it follows that $\phi_3(Z_1 \cup Z_2), \phi_3(Z_3), \ldots, \phi_3(Z_k)$ are all distinct. Hence we may assume that $\phi_1(u) = \phi_3(u)$ for all $u \in A \cap B$. But then setting $\phi(u) = \phi_1(u)$ $(u \in A)$, $\phi(u) = \phi_3(u)$ $(u \in B)$ defines a 5-colouring of G, a contradiction. ∎

If $Z_1, \ldots, Z_k \subseteq V(G)$ are disjoint, we say that $Z_1, \ldots, Z_k$ is *strongly feasible* (*via* $X_1, \ldots, X_k$) if there are disjoint fragments $X_1, \ldots, X_k$ with $Z_i \subseteq X_i$ $(1 \leq i \leq k)$ such that for $1 \leq i \leq k$, if $|Z_i| = 3$ then $G \mid X_i$ contains a triad with set of feet Z_i.

(7.7) *Under the hypothesis of* (7.6), *if* (7.6)(i) *holds, then*

(i) *if* Z_1Z_2 *are adjacent, then* $Z_1, Z_2, \ldots, Z_k$ *is infeasible in* $G \mid A$

(ii) *if* $|Z_1| = |Z_2| = 1$, *then* $Z_1 \cup Z_2, Z_3, \ldots, Z_k$ *is infeasible in* $G \mid A$

(iii) *if* $|Z_1 \cup Z_2| = 3$, *then* $Z_1 \cup Z_2, Z_3, \ldots, Z_k$ *is not strongly feasible in* $G \mid A$.

In each case the proof is clear.

(7.8) *If* v_1v_2, v_1v_3, v_1v_4 *are all adjacent then* v_5v_6 *are adjacent.*

Proof. Suppose not. Since $G \mid A \cap B$ has no circuit of length 4 by (6.7), each of v_5 and v_6 is adjacent to at most one of v_2, v_3, v_4. We may therefore assume that v_4v_5 and v_4v_6 are not adjacent. Hence by (7.5), v_2v_3 are not adjacent.

(1) $\{v_1\}, \{v_2, v_3\}, \{v_4, v_5, v_6\}$ *is infeasible in* $G \mid A$ *and in* $G \mid B$.

For suppose that $\{v_1\}, \{v_2, v_3\}, \{v_4, v_5, v_6\}$ is feasible in $G \mid A$, say. Let $Z_1 = \{v_2\}, Z_2 = \{v_3\}$, $Z_3 = \{v_4, v_5, v_6\}$, $Z_4 = \{v_1\}$. By (7.5), Z_2Z_3 are adjacent. By (7.7)(ii) there exist disjoint fragments $X_1, \ldots, X_4$ of $G \mid A$ with $Z_i \subseteq X_i$ $(1 \leq i \leq 4)$ such that X_1X_2 are adjacent. Moreover, there exist disjoint fragments $Y_1, \ldots, Y_4$ of $G \mid B$ with $Z_i \subseteq Y_i$ $(1 \leq i \leq 4)$ such that Y_1Y_3 are adjacent, by the 6-connectivity of G. This contradicts (7.6).

(2) *For* $i = 2, 3$, v_i *is not adjacent to both* v_5 *and* v_6.

Suppose that v_2v_5 and v_2v_6 are both adjacent, say. Then $\{v_3, v_5, v_6\}$ is stable and so by (7.5), v_2v_4 are not adjacent. By (1) and (7.3), $\{v_1\}, \{v_2, v_4\}, \{v_3, v_5, v_6\}$ is feasible in both $G \mid A$ and $G \mid B$, contrary to (7.4). This proves (2).

Now by (6.7), not both v_2v_4 and v_3v_4 are adjacent, and so we may assume that v_3v_4 are not adjacent. By (2), we may also assume (exchanging v_5 and v_6 if necessary) that v_2v_6 and v_3v_5 are not adjacent. By (1) and (7.3), $\{v_1\}, \{v_2, v_6\}$, $\{v_3, v_4, v_5\}$ is feasible in both $G \mid A$ and $G \mid B$, and by (7.5) $\{v_2, v_6\}$, $\{v_3, v_4, v_5\}$ are adjacent, contrary to (7.5). ∎

(7.9) $G \mid A \cap B$ *has maximum valency* ≤ 3.

Proof. Suppose that $v_1v_2, v_1v_3, v_1v_4, v_1v_5$ are all adjacent. By (7.8), v_5v_6 and v_4v_6 are adjacent, contrary to (6.7). ∎

(7.10) *If* $\{v_1, v_2, v_3\}$ *is a 3-clique then so is* $\{v_4, v_5, v_6\}$.

Proof. Suppose that v_1v_2, v_2v_3, v_1v_3 are all adjacent and v_4v_5 are not. By (7.5) we may assume that v_5v_6 are adjacent. Since $G \mid A \cap B$ has no circuits of length 4 and has maximum valency ≤ 3, we may assume that there are no edges between $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ except possibly v_1v_4, v_2v_5 and v_3v_6.

(1) *We may assume that* $\{v_1\}, \{v_2, v_6\}$, $\{v_3, v_4, v_5\}$ *and* $\{v_2\}, \{v_1, v_6\}, \{v_3, v_4, v_5\}$ *are infeasible in* $G \mid A$.

For suppose that at least one of them is feasible in $G \mid A$, and also at least one is feasible in $G \mid B$. By (7.4) neither of these partitions is feasible in both $G \mid A$ and $G \mid B$, and so we may assume that $\{v_1\}, \{v_2, v_6\}$, $\{v_3, v_4, v_5\}$ is feasible in $G \mid A$ and $\{v_2\}, \{v_1, v_6\}$, $\{v_3, v_4, v_5\}$ is feasible in $G \mid B$. Let $Z_1 = \{v_6\}$, $Z_2 = \{v_2\}$, $Z_3 = \{v_1\}$, $Z_4 = \{v_3, v_4, v_5\}$; then by (7.7)(ii), (7.6) is contradicted. This proves (1).

By (6.9), there is an octopus $\{X_1, \ldots, X_8\}$ in $G \mid A$ with base v_3, v_4, v_5 with $v_i \in X_i$ $(1 \leq i \leq 6)$. By exchanging X_7 and X_8, we may assume that X_6X_8 are adjacent. By (1), X_1X_8 are not adjacent, and so X_1X_7 are adjacent; and similarly X_2X_7 are adjacent. By (6.6) there is a 4-cluster $\{Y_1, Y_2, Y_4, Y_6\}$ of $G \mid (B - \{v_3, v_5\})$ with $v_i \in Y_i$ $(i = 1, 2, 4, 6)$. But then

$$\{X_1 \cup Y_1, X_2 \cup Y_2, X_3 \cup X_8, X_4 \cup Y_4, X_5 \cup X_7, X_6 \cup Y_6\}$$

is a 6-cluster in G, a contradiction. ∎

(7.11) $G \mid A \cap B$ *has no triangle.*

Proof. Suppose that $\{v_1, v_2, v_3\}$ is a 3-clique. Then by (7.10), $\{v_4, v_5, v_6\}$ is a 3-clique. By (7.9) and (6.7), we may assume that there is no edge between $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ except possibly v_1v_4. Now by (7.2), $\{v_1\}, \{v_4\}$, $\{v_2, v_5\}, \{v_3, v_6\}$ is feasible in both $G \mid A$ and $G \mid B$, and so by (7.4), v_1v_4 are not adjacent. Hence $G \mid A \cap B$ is the disjoint union of two triangles.

We claim that in $G \mid A$ there are three disjoint paths from $\{v_1, v_2, v_3\}$ to $\{v_4, v_5, v_6\}$. For if not, then there is a (≤ 2)-separation (X, Y) of $G \mid A$ with $v_1, v_2, v_3 \in X$ and $v_4, v_5, v_6 \in Y$. Then $(X, B \cup Y)$ is a separation of G of order

$$|X \cap Y| + |X \cap (B - Y)| \leq |X \cap Y| + |(A \cap B) - Y| \leq 5$$

and so $B \cup Y = V(G)$; and similarly $B \cup X = V(G)$. Since $|X \cap Y| \leq 2$, it follows that $|A - B| \leq 2$, contrary to (6.3). This proves that there are three disjoint paths of G from $\{v_1, v_2, v_3\}$ to $\{v_4, v_5, v_6\}$; and therefore from the symmetry we may assume that $\{v_1, v_4\}, \{v_2, v_5\}, \{v_3, v_6\}$ is feasible in $G \mid A$. But $\{v_1\}, \{v_4\}, \{v_2, v_5\}, \{v_3, v_6\}$ is feasible in $G \mid B$ by (7.2), contrary to (7.6) and (7.7)(i),(ii), taking $Z_1 = \{v_1\}, Z_2 = \{v_4\}$, $Z_3 = \{v_2, v_5\}, Z_4 = \{v_3, v_6\}$. This completes the proof. ∎

(7.12) $G \mid A \cap B$ *has no circuit.*

Proof. From (7.11), $G \mid A \cap B$ has no triangle, and from (6.7), it has no circuit of length 4. Suppose that $\{v_1, v_2, v_3, v_4, v_5\} \subseteq A \cap B$ is the vertex set of a circuit of length 5, numbered in order. By (6.7) and (7.11), v_6 has valency ≤ 1 in $G \mid A \cap B$ and $G \mid \{v_1, \ldots, v_5\}$ has no more edges. Suppose first that v_5v_6 are adjacent. From (7.4), $\{v_5\}, \{v_2, v_4\}$, $\{v_1, v_3, v_6\}$ is infeasible in one of $G \mid A, G \mid B$, say $G \mid A$.

By (6.9) there is an octopus $\{X_1,\ldots,X_8\}$ in $G\,|\,A$ on $A\cap B$ with base v_1,v_3,v_4 with $v_i \in X_i$ $(1\leq i\leq 6)$. From the symmetry we may assume that X_2X_7 are adjacent. Then X_6X_8 are not adjacent since $\{v_5\},\{v_2,v_4\},\{v_1,v_3,v_6\}$ is infeasible in $G\,|\,A$, and so X_6X_7 are adjacent. By (6.6) there is a 4-cluster $\{Y_2,Y_3,Y_4,Y_5\}$ in $G\,|\,(B-\{v_1,v_6\})$ with $v_i\in Y_i\,(i=2,3,4,5)$. Then

$$\{X_1\cup X_8, X_2\cup Y_2, X_3\cup Y_3, X_4\cup Y_4, X_5\cup Y_5, X_6\cup X_7\}$$

is a 6-cluster in G, a contradiction.

This proves that v_5v_6 are not adjacent, and so v_6 has valency 0 in $G\,|\,A\cap B$. By a *crux* we mean a partition Z_1,Z_2,Z_3 of $\{v_1,\ldots,v_6\}$ such that $|Z_1|=1$, $|Z_2|=2, |Z_3|=3$, and Z_1,Z_2,Z_3 are all stable. Necessarily, $v_6\in Z_3$. There are ten cruces in total. For $1\leq i\leq 5$, there are two cruces Z_1,Z_2,Z_3 with $Z_1=\{v_i\}$, and one of them is feasible in $G\,|\,A$, by (7.3). Thus at least five cruces are feasible in $G\,|\,A$, and at least five in $G\,|\,B$. On the other hand, no crux is feasible in both $G\,|\,A$ and $G\,|\,B$ by (7.4), and so for each i $(1\leq i\leq 5)$ there is exactly one crux Z_1,Z_2,Z_3 with $|Z_1|=\{v_i\}$ feasible in $G\,|\,A$. Moreover, every crux is feasible in exactly one of $G\,|\,A$, $G\,|\,B$.

If Z_1,Z_2,Z_3 is a crux its *mate* is the unique crux Z_1',Z_2',Z_3', with $Z_3'=Z_3$ and $Z_1'\neq Z_1$. This provides an involution among the set of cruces, and since an odd number of cruces are feasible in $G\,|\,A$, there is one feasible in $G\,|\,A$ such that its mate is infeasible in $G\,|\,A$. We may therefore assume that $\{v_1\},\{v_2,v_4\},\{v_3,v_5,v_6\}$ is feasible in $G\,|\,A$, and its mate $\{v_2\},\{v_1,v_4\},\{v_3,v_5,v_6\}$ is infeasible in $G\,|\,A$. Consequently, the latter is feasible in $G\,|\,B$, contrary to (7.6) and (7.7)(ii), taking $Z_1=\{v_4\}$, $Z_2=\{v_2\}, Z_3=\{v_1\}$, $Z_4=\{v_3,v_5,v_6\}$.

This proves that $G\,|\,A\cap B$ has no circuit of length 5. To complete the proof, we suppose that $G\,|\,A\cap B$ has a circuit of length 6; and then, by (7.11) and (6.7), it has no more edges. Let $A\cap B=\{v_1,\ldots,v_6\}$ numbered in order on the circuit. By (7.4), $\{v_1,v_3,v_5\}$, $\{v_2,v_4,v_6\}$ is infeasible in one of $G\,|\,A, G\,|\,B$, say $G\,|\,A$. By (6.9) there is an octopus $\{X_1,\ldots,X_8\}$ in $G\,|\,A$ on $A\cap B$ with base v_2,v_4,v_6, with $v_i\in X_i$ $(1\leq i\leq 6)$.

Now not all X_1,X_3,X_5 are adjacent to X_7 since $\{v_1,v_3,v_5\},\{v_2,v_4,v_6\}$ is not feasible in $G\,|\,A$, and similarly they are not all adjacent to X_8. Thus we may assume that X_1X_7,X_3X_8,X_5X_8 are adjacent. By (6.6) there is a 4-cluster $\{Y_1,Y_2,Y_3,Y_5\}$ in $G\,|\,(B-\{v_4,v_6\})$ with $v_i\in Y_i$ $(i=1,2,3,5)$. But then

$$\{X_1\cup Y_1, X_2\cup Y_2, X_3\cup Y_3, X_4\cup X_7, X_5\cup Y_5, X_6\cup X_8\}$$

is a 6-cluster in G, a contradiction. ∎

(7.13) *$G\,|\,A\cap B$ has maximum valency ≤ 2.*

Proof. By (7.9), $G\,|\,A\cap B$ has maximum valency ≤ 3. Suppose that v_1 is adjacent to v_2,v_3,v_4. Then by (7.8), v_5v_6 are adjacent. By (7.3), one of

$$\{v_1\},\{v_2,v_5\},\{v_3,v_4,v_6\}$$
$$\{v_1\},\{v_3,v_5\},\{v_2,v_4,v_6\}$$
$$\{v_1\},\{v_4,v_5\},\{v_2,v_3,v_6\}$$

is feasible in both $G \mid A$ and $G \mid B$, and so by (7.4) one of these sets is not stable. From the symmetry and (7.12) we may assume that v_2v_5 are adjacent; and then by (7.12) $G \mid A \cap B$ has no more edges. By (7.3), one of

$$\{v_1\}, \{v_4, v_5\}, \{v_2, v_3, v_6\}$$
$$\{v_1\}, \{v_3, v_5\}, \{v_2, v_4, v_6\}$$
$$\{v_1\}, \{v_2, v_6\}, \{v_3, v_4, v_5\}$$

is feasible in both $G \mid A$ and $G \mid B$, contrary to (7.4). ∎

(7.14) *$G \mid A \cap B$ has ≥ 3 edges.*

Proof. By (7.5) no vertex of $G \mid A \cap B$ meets all its edges, and so we may assume that v_1v_2 are adjacent and v_3v_4 are adjacent, and $G \mid A \cap B$ has no more edges.

(1) *$\{v_1, v_3\}, \{v_2, v_4, v_5, v_6\}$ is infeasible in $G \mid A$ and in $G \mid B$.*

For let $Z_1 = \{v_1\}$, $Z_2 = \{v_3\}$, $Z_3 = \{v_2, v_4, v_5, v_6\}$; the claim follows from (7.6), (7.7)(i) and (7.7)(ii). ((7.6)(ii) does not hold since Z_1Z_3 are adjacent.)

Now by (7.3), one of

$$\{v_6\}, \{v_1, v_3\}, \{v_2, v_4, v_5\}$$
$$\{v_6\}, \{v_2, v_3\}, \{v_1, v_4, v_5\}$$
$$\{v_6\}, \{v_1, v_4\}, \{v_2, v_3, v_5\}$$

is feasible in both $G \mid A$ and $G \mid B$. (This does *not* contradict (7.4).) From the symmetry we may assume the first. Consequently, there are disjoint fragments X_1, X_2 of $G \mid A$ with $v_1, v_3 \in X_1$ and $v_2, v_4, v_5 \in X_2$. Choose X_1, X_2 maximal. Then $v_6 \in X_1 \cup X_2$, and by (1) $v_6 \notin X_2$. Thus $v_6 \in X_1$. We have therefore proved that $\{v_1, v_3, v_6\}, \{v_2, v_4, v_5\}$ is feasible in $G \mid A$. But by symmetry it is also feasible in $G \mid B$, contrary to (7.4). ∎

(7.15) *$G \mid A \cap B$ has ≥ 4 edges.*

Proof. Suppose it has only three. Suppose that it has ≥ 2 vertices of valency 0; then we may assume its edges are v_1v_2, v_2v_3, v_3v_4. Then $\{v_1, v_4, v_5, v_6\}$ is stable and v_2v_3 are adjacent, contrary to (7.5).

Thus $G \mid A \cap B$ has at most one vertex of valency 0. Suppose it has one. Then we may assume its edges are v_1v_2, v_3v_4, v_4v_5. By (7.3) one of

$$\{v_4\}, \{v_1, v_3\}, \{v_2, v_5, v_6\}$$
$$\{v_4\}, \{v_1, v_5\}, \{v_2, v_3, v_6\}$$
$$\{v_4\}, \{v_2, v_3\}, \{v_1, v_5, v_6\}$$

is feasible in both $G \mid A$ and $G \mid B$, contrary to (7.4).

Hence $G \mid A \cap B$ has minimum valency ≥ 1; and hence we may assume its edges are v_1v_2, v_3v_4, and v_5v_6.

(1) *If* $\{v_1,v_3,v_5\}$, $\{v_2,v_4,v_6\}$ *is strongly feasible in* $G\mid A$, *then* $\{v_5\}$, $\{v_1,v_3\}$, $\{v_2,v_4,v_6\}$ *is infeasible in* $G\mid B$.

For take $Z_1=\{v_5\}, Z_2=\{v_1,v_3\}$, $Z_3=\{v_2,v_4,v_6\}$; the claim follows from (7.6) and (7.7)(iii).

By (6.9) there is an octopus $\{X_1,\dots,X_8\}$ in $G\mid A$ on $A\cap B$ with base v_1,v_3,v_5, and an octopus $\{Y_1,\dots,Y_8\}$ in $G\mid B$ on $A\cap B$ with base v_1,v_3,v_5. From the symmetry we may assume that X_2X_7 and Y_2Y_7 are adjacent.

(2) *Not both* X_4X_7 *and* X_6X_7 *are adjacent.*

For suppose they are. Then $\{v_1,v_3,v_5\},\{v_2,v_4,v_6\}$ is strongly feasible in $G\mid A$. If Y_4Y_7 are adjacent, then $\{v_6\},\{v_2,v_4\}$, $\{v_1,v_3,v_5\}$ is feasible in $G\mid B$ contrary to (1). Thus Y_4Y_8 and similarly Y_6Y_8 are adjacent. But then $\{v_2\},\{v_4,v_6\}$, $\{v_1,v_3,v_5\}$ is feasible in $G\mid B$, contrary to (1).

We may therefore assume that X_6X_7 are not adjacent, and hence X_6X_8 are adjacent. Hence there is symmetry between X_7 and X_8 (exchanging v_1,v_2 with v_5,v_6 and exchanging Y_7,Y_8 if necessary), and so we may assume that X_4X_7 are adjacent. Now $\{v_1,v_3,v_6\},\{v_2,v_4,v_5\}$ is strongly feasible in $G\mid A$, and so Y_4Y_7 are not adjacent, since $\{v_6\},\{v_1,v_3\}$, $\{v_2,v_4,v_5\}$ is infeasible in $G\mid B$ by (1); and Y_6Y_8 are not adjacent, since $\{v_4\},\{v_2,v_5\},\{v_1,v_3,v_6\}$ is infeasible in $G\mid B$ by (1). Hence Y_4Y_8 and Y_6Y_7 are adjacent. But then $\{v_2\},\{v_4,v_5\},\{v_1,v_3,v_6\}$ is feasible in $G\mid B$ contrary to (1). ∎

(7.16) $G\mid A\cap B$ *is a 5-edge path.*

Proof. If not, then from (7.15), (7.12) and (7.13), $G\mid A\cap B$ is the disjoint union of two paths. There are three cases depending on the lengths of these paths.

First, we assume that v_i is adjacent to v_{i+1} for $i=1,2,3,4$, and v_6 has valency 0.

(1) $\{v_1,v_3,v_5,v_6\},\{v_2,v_4\}$ *is infeasible in* $G\mid A$ *and in* $G\mid B$.

For let $Z_1=\{v_2\}, Z_2=\{v_4\}$, $Z_3=\{v_1,v_3,v_5,v_6\}$; the claim follows from (7.6) and (7.7)(ii).

By (7.4), $\{v_3\},\{v_2,v_5\},\{v_1,v_4,v_6\}$ is infeasible in one of $G\mid A$ and $G\mid B$, say in $G\mid B$. Hence by (7.3), $\{v_3\},\{v_1,v_4\},\{v_2,v_5,v_6\}$ is feasible in $G\mid B$ and hence not in $G\mid A$, by (7.4).

There is still symmetry between A and B (exchanging v_1 and v_5). By (7.4), $\{v_1,v_3,v_5\}$, $\{v_2,v_4,v_6\}$ is infeasible in one of $G\mid A$ and $G\mid B$, say in $G\mid A$. By (6.9) there is an octopus $\{X_1,\dots,X_8\}$ in $G\mid A$ on $A\cap B$ with base v_1,v_3,v_5, with $v_i\in X_i$ $(1\le i\le 6)$, and there is an octopus $\{Y_1,\dots,Y_8\}$ in $G\mid B$ on $A\cap B$ with base v_1,v_3,v_5, with $v_i\in Y_i$ $(1\le i\le 6)$. We may assume that X_2X_7 are adjacent. Now either X_6X_7 or X_6X_8 are adjacent, and yet both

$$\{v_1,v_3,v_5\},\{v_2,v_4,v_6\}$$
$$\{v_1,v_3,v_5,v_6\},\{v_2,v_4\}$$

are infeasible in $G\mid A$, and so X_4X_7 are not adjacent. Hence X_4X_8 are adjacent. Since $\{v_3\},\{v_1,v_4\},\{v_2,v_5,v_6\}$ is infeasible in $G\mid A$, it follows that X_6X_7 are not adjacent, and so X_6X_8 are adjacent.

We may assume that Y_2Y_7 are adjacent. If Y_6Y_8 are adjacent, then either Y_4Y_7 are adjacent or Y_4Y_8 are adjacent, and yet

$$\{v_1, v_3, v_5, v_6\}, \{v_2, v_4\}$$
$$\{v_3\}, \{v_2, v_5\}, \{v_1, v_4, v_6\}$$

are both infeasible in $G \mid B$, which is impossible. Thus Y_6Y_8 are not adjacent, and so Y_6Y_7 are adjacent. But then

$$\{X_1 \cup Y_1 \cup Y_8, X_2 \cup Y_2 \cup X_7, X_3 \cup Y_3 \cup X_4 \cup Y_4, X_5 \cup Y_5, X_6 \cup Y_6 \cup Y_7, X_8\}$$

is a 6-cluster in G (recall that X_7X_8 are adjacent), a contradiction. This concludes the first case.

Now we assume that v_iv_{i+1} are adjacent for $i = 1, 2, 3, 5$. By (7.4), we may assume that $\{v_3\}$, $\{v_2,v_5\},\{v_1,v_4,v_6\}$ is infeasible in $G \mid A$. By (7.3), $\{v_3\},\{v_2,v_6\}$, $\{v_1,v_4,v_5\}$ is feasible in $G \mid A$, and hence not in $G \mid B$, by (7.4). By (7.3), $\{v_3\},\{v_2,v_5\}$, $\{v_1,v_4,v_6\}$ is feasible in $G \mid B$. Suppose that $\{v_2\},\{v_3,v_5\}$, $\{v_1,v_4,v_6\}$ is feasible in $G \mid A$. Take $Z_1 = \{v_5\}$, $Z_2 = \{v_3\}$, $Z_3 = \{v_2\}$, $Z_4 = \{v_1,v_4,v_6\}$; then (7.6) is contradicted. This proves that $\{v_2\},\{v_3,v_5\}$, $\{v_1,v_4,v_6\}$ is infeasible in $G \mid A$, and, by the symmetry, $\{v_2\},\{v_3,v_6\}$, $\{v_1,v_4,v_5\}$ is infeasible in $G \mid B$.

Let $\{X_1,\ldots,X_8\}$ be an octopus on $A \cap B$ in $G \mid A$ with base v_1,v_3,v_6, with $v_i \in X_i$ $(1 \le i \le 6)$. We may assume that X_4X_7 are adjacent. Since $\{v_2\},\{v_3,v_5\}$, $\{v_1,v_4,v_6\}$ is infeasible in $G \mid A$, it follows that X_5X_8 are not adjacent, and hence X_5X_7 are adjacent.

Suppose that X_2X_7 are adjacent. By (6.6) there is a 4-cluster $\{C_1,C_2,C_4,C_6\}$ in $G \mid (B - \{v_3,v_5\})$ with $v_i \in C_i$ $(i=1,2,4,6)$. But then

$$\{X_1 \cup C_1, X_2 \cup C_2, X_3 \cup X_8, X_4 \cup C_4, X_5 \cup X_7, X_6 \cup C_6\}$$

is a 6-cluster in G, a contradiction. Thus X_2X_8 are adjacent.

Now, there is a symmetry exchanging A with B, v_5 with v_6, v_1 with v_4 and v_2 with v_3. Consequently, there is an octopus $\{Y_1,\ldots,Y_8\}$ in $G \mid B$ with base v_2,v_4,v_5, with $v_i \in Y_i$ $(1 \le i \le 6)$, and with Y_1Y_7, Y_6Y_7, Y_3Y_8 adjacent. But then

$$\{X_1 \cup Y_1 \cup X_7, X_2 \cup Y_2, X_3 \cup Y_3, X_4 \cup Y_4 \cup Y_7, X_5 \cup Y_5 \cup Y_8, X_6 \cup Y_6 \cup X_8\}$$

is a 6-cluster, a contradiction. This concludes the second case.

In the third case, we assume that v_iv_{i+1} are adjacent for $i = 1,2,4,5$. By (7.4), $\{v_5\},\{v_2,v_4\}$, $\{v_1,v_3,v_6\}$ is infeasible in one of $G \mid A, G \mid B$, say $G \mid A$. By (7.3), $\{v_5\},\{v_2,v_6\}$, $\{v_1,v_3,v_4\}$ is feasible in $G \mid A$, and hence not in $G \mid B$.

Let $\{X_1,\ldots,X_8\}$ be an octopus in $G \mid A$ on $A \cap B$ with base v_1,v_3,v_4 with $v_i \in X_i$ $(1 \le i \le 6)$. We may assume that X_2X_7 are adjacent. Since

$$\{v_5\}, \{v_2, v_4\}, \{v_1, v_3, v_6\}$$

is infeasible in $G \mid A$ it follows that X_6X_8 are not adjacent, and so X_6X_7 are adjacent. Suppose that X_5X_8 are adjacent; let $\{C_2,C_3,C_4,C_5\}$ be a 4-cluster in $G \mid (B - \{v_1,v_6\})$ with $v_i \in C_i$ $(i=2,3,4,5)$, and then

$$\{X_1 \cup X_8, X_2 \cup C_2, X_3 \cup C_3, X_4 \cup C_4, X_5 \cup C_5, X_6 \cup X_7\}$$

is a 6-cluster, a contradiction. Hence X_5X_8 are not adjacent, and so X_5X_7 are adjacent.

Let $\{Y_1,\ldots,Y_8\}$ be an octopus in $G\,|\,B$ on $A\cap B$ with base v_1,v_2,v_6, with $v_i \in Y_i$ $(1\le i\le 6)$. We may assume that Y_3Y_7 are adjacent. Then Y_4Y_7 are not adjacent, since $\{v_5\},\{v_2,v_6\},\{v_1,v_3,v_4\}$ is infeasible in $G\,|\,B$. Hence, Y_4Y_8 are adjacent. But then

$$\{X_1\cup Y_1, X_2\cup Y_2, X_3\cup Y_3\cup X_8, X_4\cup Y_4\cup Y_8, X_5\cup Y_5\cup X_7, X_6\cup Y_6\cup Y_7\}$$

is a 6-cluster in G, a contradiction. ∎

The remaining case, when $G\mid A\cap B$ is a 5-edge path, unfortunately resists attack by these methods, and we need another technique, which we develop in the next two sections.

8. The infeasible partitions

Now we use the methods of the last section to learn what we can about the case left open by (7.16). We need the following lemma.

(8.1) *Let X,Y,Z be finite sets of integers, with $|X|,|Y|,|Z|\ge 3$. Then either*

(i) *for one of X,Y,Z, say X, there are two members $x_1,x_2\in X$ and $y\in Y$ and $z\in Z$ with $x_1<y<x_2$ and $x_1<z<x_2$, or*

(ii) *for some integer n, let I, J be the sets of integers $\le n$ and $\ge n$ respectively; then both I and J include one of X, Y, Z, and X, Y, Z are all subsets of one of I, J.*

Proof. Let x_1 be the smallest member of X, and x_2 the largest; and define y_1,y_2,z_1,z_2 similarly. We may assume that $x_2-x_1\ge y_2-y_1, z_2-z_1$. Let $A=\{n : x_1<n<x_2\}$. If $A\cap Y\ne\emptyset$ and $A\cap Z\ne\emptyset$ then (i) holds, and so we may assume that $A\cap Y=\emptyset$. If $y_1<x_2$ and $y_2>x_1$ then it follows that $y_1\le x_1$ (since $y_1\notin A$) and similarly $y_2\ge x_2$; and since $x_2-x_1\ge y_2-y_1$, we deduce that $Y=\{y_1,y_2\}$, a contradiction since $|Y|\ge 3$. Thus, either $x_2\le y_1$ or $y_2\le x_1$, and from the symmetry we may assume that $x_2\le y_1$. If $x_2\le z_1$, then (ii) holds, and so we may assume that $z_1<x_2$, and similarly $y_1<z_2$. But then (i) holds (with z_1,x_2,y_1,z_2). ∎

We use (8.1) to prove the following.

(8.2) *Let G be a graph and let $v_1,\ldots,v_5\in V(G)$ be distinct. Suppose that*

(i) *there is a circuit of G containing v_1,v_2,v_3, disjoint from a path of G with ends v_4,v_5,*

(ii) *$\{v_1,v_2,v_3\}$ is stable,*

(iii) *there is no 5-separation (A,B) of G with $\{v_1,\ldots,v_5\}\subseteq A$, $|A|>5$ and $|B-A|\ge 2$, and*

(iv) *there is no (≤ 4)-separation (A,B) of G with $\{v_1,\ldots,v_5\}\subseteq A$ and $|B-A|\ge 1$.*

Then $\{v_1, v_2, v_3\}$, $\{v_4, v_5\}$ *is strongly feasible in* G.

Proof. Let C be a circuit of G with $v_1, v_2, v_3 \in V(C)$, and let P be a path of G with ends v_4, v_5, with $P \cap C$ null. Let the path of C between v_1 and v_2 not containing v_3 be C_{12}, and define C_{13}, C_{23} similarly. Since $\{v_1, v_2, v_3\}$ is stable, it follows that $|E(C_{12})| \geq 2$, and so there is a unique component H_{12} of $G \setminus (V(P) \cup \{v_1, v_2, v_3\})$ containing $C_{12} \setminus \{v_1, v_2\}$. Define H_{13}, H_{23} similarly, and choose P and C so that $H_{12} \cup H_{13} \cup H_{23}$ is maximal.

(1) *We may assume that* v_3 *has no neighbour in* $V(H_{12})$, *and similarly for* $v_2, V(H_{13})$ *and for* $v_1, V(H_{23})$; *and in particular* H_{12}, H_{13}, H_{23} *are all distinct.*

For if $V(H_{12})$ contains a neighbour of v_3, then $V(H_{12}) \cup \{v_1, v_2, v_3\}$ contains a triad with feet v_1, v_2, v_3, and so the theorem is true.

Let X_3 be the set of vertices in $V(P)$ with a neighbour in $V(H_{12})$, and define X_2, X_1, similarly for $V(H_{13}), V(H_{23})$.

(2) $|X_1|, |X_2|, |X_3| \geq 3$.

For if $|X_3| \leq 2$ say, let $B = V(H_{12}) \cup \{v_1, v_2\} \cup X_3$ and $A = V(G) - V(H_{12})$; then $A \cap B = \{v_1, v_2\} \cup X_3$ and so (A, B) is a (≤ 4)-separation of G; but $\{v_1, \ldots, v_5\} \subseteq A$, and

$$B - A = V(H_{12}) \neq \emptyset$$

contrary to hypothesis (iv).

(3) *Let* $v \in V(P)$, *and let* P_1, P_2 *be the subpaths of* P *between* v *and* v_4, v_5 *respectively. Then it is not true that* $X_1, X_2 \subseteq V(P_1)$ *and* $X_3 \subseteq V(P_2)$.

For suppose this is true. We may assume that $v \in X_3$. Let

$$A = V(H_{12}) \cup \{v_1, v_2, v_3, v_4\} \cup V(P_2),$$
$$B = V(H_{13} \cup H_{23}) \cup \{v_1, v_2, v_3\} \cup V(P_1).$$

Since $\{v_1, \ldots, v_5\} \subseteq A$, $|A| > 5$, $|B - A| \geq 2$ and $A \cap B = \{v, v_1, v_2, v_3, v_4\}$, it follows from hypothesis (iii) that there is a path Q of G from A to B with $V(Q) \cap A \cap B = \emptyset$; choose Q minimal, with ends $a \in A - B$ and $b \in B - A$ say. By the minimality of Q, $V(Q) \cap (A \cup B) = \{a, b\}$. If $a \in V(H_{12})$ then

$$V(Q) \cap (V(P) \cup \{v_1, v_2, v_3\}) \subseteq V(Q) \cap (A \cup B - \{a\}) = \{b\},$$

and hence $Q \setminus b \subseteq H_{12}$; but then $b \in B - A$ has a neighbour in $V(H_{12})$, and hence $b \in X_3 \subseteq A$, a contradiction. Thus $a \notin V(H_{12})$, and similarly $b \notin V(H_{13} \cup H_{23})$. Since $a, b \notin A \cap B$, it follows that $a \in V(P_2) - \{v\}$, and similarly $b \in V(P_1) - \{v, v_4\}$. Let P' be obtained from $P \cup Q$ by deleting the edges and vertices of P strictly between a and b; then $P' \cap C$ is null, and $V(H_{12}), V(H_{13}), V(H_{23})$ are all disjoint from $V(P' \cup C)$, and $v \notin V(P' \cup C)$ and has a neighbour in $V(H_{12})$ (since $v \in X_3$). This contradicts the choice of P, C. This proves (3).

From (2), (3) and (8.1), we may assume that there exist $a, b \in X_3$, such that if R denotes the subpath of P between a and b, then for $i = 1, 2$ there exists $x_i \in X_i \cap V(R) - \{a, b\}$. Let P' be a path of G between v_4 and v_5 obtained from P by replacing R by a path from a to b with vertex set in $\{a, b\} \cup V(H_{12})$. Let T be the union of C_{13}, a minimal path in $\{x_2\} \cup V(H_{13})$ between $V(C_{13}) - \{v_1, v_3\}$ and x_2, the subpath of P between x_2 and x_1, a minimal path in $\{x_1\} \cup V(H_{23})$ between x_1 and some $u \in V(C_{23}) - \{v_2, v_3\}$, and the subpath of C_{23} between u and v_2; then T is a triad with feet v_1, v_2, v_3, disjoint from P'. Hence the result is true. ∎

(8.3) *Let G be a graph, let $v_1, \ldots, v_5$ be distinct, and let C be a circuit of $G \setminus \{v_4, v_5\}$ with $v_1, v_2 \in V(C)$. Let R_3 be a path of $G \setminus \{v_1, v_2, v_4, v_5\}$ from v_3 to $h_3 \in V(C)$, with no vertex in C except h_3. Let C_{12} be the path of C between v_1 and v_2 not containing h_3, and for $i = 1, 2$, let C_{i3} be the path of C between v_i and h_3 not containing v_{3-i}. Let H_i be the component of $G \setminus V(C \cup R_3)$ containing v_i for $i = 4, 5$, and suppose that for $i = 4, 5$ both $V(C_{13}) - \{v_1, h_3\}$ and $V(C_{23}) - \{v_2, h_3\}$ contain a vertex with a neighbour in $V(H_i)$. Suppose also that*

(i) *$\{v_1, v_2, v_3\}$ is stable,*

(ii) *there is no 5-separation (A, B) of G with $\{v_1, \ldots, v_5\} \subseteq A, |A| > 5$ and $|B - A| \geq 2$, and*

(iii) *there is no (≤ 4)-separation (A, B) of G with $\{v_1, \ldots, v_5\} \subseteq A$ and $|B - A| \geq 1$.*

Then $\{v_1, v_2, v_3\}, \{v_4, v_5\}$ is strongly feasible.

Proof. For i = 4, 5, choose $h_i \in V(H_i)$ such that there are three paths P_i, Q_i, R_i of G, mutually disjoint except for h_i, where

(i) P_i is from h_i to $V(C_{13}) - \{v_1, h_3\}$, and has no vertex in $C \cup R_3$ except its end $p_i \in V(C_{13}) - \{v_1, h_3\}$

(ii) Q_i is from h_i to $V(C_{23}) - \{v_2, h_3\}$, and has no vertex in $C \cup R_3$ except its end $q_i \in V(C_{23}) - \{v_2, h_3\}$

(iii) R_i is from h_i to v_i, and is disjoint from $C \cup R_3$.

Moreover, choose C, h_4, h_5 etc., so that $|E(R_3)| + |E(R_4)| + |E(R_5)|$ is minimum.

(1) *We may assume that $(P_4 \cup Q_4 \cup R_4) \cap (P_5 \cup Q_5 \cup R_5) \subseteq C$.*

For otherwise there is a path from v_4 to v_5 disjoint from $C \cup R_3$; and hence if $E(R_3) \neq \emptyset$ this path is disjoint from a triad with feet v_1, v_2, v_3 as required, and if $E(R_3) = \emptyset$ the result follows from (8.2).

Let

$$A = V(C_{12} \cup R_3 \cup R_4 \cup R_5)$$

$$B = V(C_{13} \cup C_{23} \cup P_4 \cup Q_4 \cup P_5 \cup Q_5).$$

Then $A \cap B = \{v_1, v_2, h_3, h_4, h_5\}$ and $|B - A| \geq 2$ (since C_{13}, C_{23} both have internal vertices) and $|A| \geq 6$ (since $|V(C_{12})| \geq 3$), and so from the hypothesis there is a path Q of G from A to B with $V(Q) \cap A \cap B = \emptyset$. Choose Q minimal, with ends $a \in A - B$ and $b \in B - A$, and hence with $V(Q) \cap (A \cup B) = \{a, b\}$. From the symmetry we may assume that $a \in V(C_{12} \cup R_3 \cup R_4) - \{v_1, v_2, h_3, h_4\}$ and $b \in V(C_{13} \cup P_4 \cup P_5) - \{v_1, h_3, h_4, h_5\}$.

Suppose first that $a \in V(C_{12})$. Then

$$Q \cup C_{12} \cup (P_4 \setminus h_4) \cup (P_5 \setminus h_5) \cup C_{13} \cup R_3$$

contains a triad with feet v_1, v_2, v_3, disjoint from the path between v_4 and v_5 in $R_4 \cup Q_4 \cup (C_{23} \setminus \{v_2, h_3\}) \cup Q_5 \cup R_5$, and so the result is true. Now suppose that

$a \in V(R_4) - \{h_4\}$. Then since $b \in V(C_{13} \cup P_4 \cup P_5)$, this contradicts the minimality of $|E(R_3)| + |E(R_4)| + |E(R_5)|$, as we see by replacing an appropriate subpath of P_4 by Q. Finally, suppose that $a \in V(R_3) - \{h_3\}$. Then we can replace C_{13} by a path between v_1 and a in

$$((C_{13} \cup R_3) \setminus h_3) \cup Q \cup (P_4 \setminus h_4) \cup (P_5 \setminus h_5),$$

replace h_3 by a, and change C_{23} and R_3 accordingly, again contrary to the minimality of $|E(R_3)| + |E(R_4)| + |E(R_5)|$. The result follows. ∎

(8.4) *Let G be a graph, and let $v_1, \ldots, v_5 \in V(G)$ be distinct, such that*

$$\{v_1\}, \{v_2, v_3\}, \{v_4, v_5\}$$
$$\{v_2\}, \{v_1, v_3\}, \{v_4, v_5\}$$
$$\{v_3\}, \{v_1, v_2\}, \{v_4, v_5\}$$
$$\{v_1, v_2, v_3\}, \{v_4, v_5\}$$

are all feasible in G. Suppose also that

(i) *$\{v_1, v_2, v_3\}$ is stable*

(ii) *there is no 5-separation (A,B) of G with $v_1, \ldots, v_5 \in A$, $|A| > 5$ and $|B - A| \geq 2$, and*

(iii) *there is no (≤ 4)-separation (A,B) of G with $v_1, \ldots, v_5 \in A$ and $|B - A| \geq 1$.*

Then $\{v_1, v_2, v_3\}, \{v_4, v_5\}$ is strongly feasible.

Proof. Let $\mathscr{P}$ be the partition $\{v_1, v_2, v_3\}$, $\{v_4, v_5\}$. Since $\mathscr{P}$ is feasible, we may assume that there is a path H of G with ends v_1, v_2 and with $v_3 \in V(H)$, and a path J of G with ends v_4, v_5, such that $H \cap J$ is null. Since $\{v_3\}, \{v_1, v_2\}$, $\{v_4, v_5\}$ is feasible, there is a path P of $G \setminus v_3$ with ends v_1, v_2 and a path Q of $G \setminus v_3$ with ends v_4, v_5, such that $P \cap Q$ is null. Choose H, J, P, Q so that $H \cup J \cup P \cup Q$ is minimal.

By an *arc* we mean a subpath of $P \cup Q$ with distinct ends both in $V(H \cup J)$ and with no edge or internal vertex in $H \cup J$.

(1) *We may assume that every arc has one end in $V(H)$ and the other in $V(J)$.*

For let R be an arc with ends a, b say. If $a, b \in V(J)$, let J' be obtained from $J \cup R$ by deleting the edges and vertices of J strictly between a and b; then H, J', P, Q contradicts the minimality of $H \cup J \cup P \cup Q$. Thus not both $a, b \in V(J)$. Similarly not both a, b belong to the subpath of H between v_1 and v_3, or to the subpath between v_2 and v_3. Consequently, if $a, b \in V(H)$ then we may assume that v_1 and a belong to one component of $H \setminus v_3$, and v_2 and b to the other. If $v_1 = a$ and $v_2 = b$ then $\mathscr{P}$ is strongly feasible by (8.2); and otherwise $\mathscr{P}$ is strongly feasible since $R \cup H$ includes a triad with feet v_1, v_2, v_3. This proves (1).

(2) *Both P and Q include arcs.*

For since P has ends v_1, v_2 and $v_3 \notin V(P)$, it follows that $P \not\subseteq H \cup J$ and so P includes an arc. Suppose that Q includes no arc; then $Q = J$, and so the arc in P has both ends in $V(H)$, contrary to (1).

Let P_1 and P_2 be the arcs in P closest in P to v_1 and to v_2 respectively; these exist by (2). Let P_i have ends v_i' and u_i, where v_i' lies in P between u_i and v_i ($i =$

1,2). Let R_i be the subpath of P between v_i and v_i'. Then $R_i \subseteq H \cup J$, and so $R_i \subseteq H$ since $v_i \in V(R_i \cap H)$. By (1), $u_1, u_2 \in V(J)$.

Let Q_4 and Q_5 be the arcs in Q closest to v_4 and to v_5 respectively; for $i = 4,5$ let Q_i have ends v_i' and u_i, where v_i' lies in Q between u_i and v_i. Let R_i be the subpath of Q between v_i and v_i'. Then $R_i \subseteq J$ $(i=4,5)$, and by (1), $u_4, u_5 \in V(H)$.

(3) *u_4 and u_5 belong to the same component of $H \setminus v_3$.*

For suppose that u_4 and v_1 belong to one component, H_1 say, and u_5 and v_2 to the other, H_2 say. For $i=1,2,4,5$, let v_i'' be the vertex of $H \cup J$ which

(i) belongs to the same component of $H \cup J$ as v_i

(ii) belongs to $V(P \cup Q)$

(iii) does not belong to $V(R_i)$

(iv) subject to (i)–(iii), is closest in $H \cup J$ to v_i.

Since $u_4 \in V(H_1)$ and $u_4 \notin V(R_1)$, it follows that v_1'' lies in H strictly between v_1' and v_3, and similarly v_2'' lies in H strictly between v_3 and v_2'. If $v_1'' \in V(P)$, let P' be obtained from P by replacing the subpath of P between v_1' and v_1'' by the subpath of H between these vertices; then $P' \cap Q$ is null, contrary to the minimality of $H \cup J \cup P \cup Q$. Thus $v_1'' \in V(Q)$ and similarly $v_2'' \in V(Q)$, $v_4'' \in V(P), v_5'' \in V(P)$. Let P' be the union of the subpath of H between v_1 and v_1'', the subpath of Q between v_1'' and v_2'', and the subpath of H between v_2'' and v_2. Let Q' be the union of the subpath of J between v_4 and v_4'', the subpath of P between v_4'' and v_5'', and the subpath of J between v_5'' and v_5. Then $P' \cap Q'$ is null, contrary to the minimality of $H \cup J \cup P \cup Q$. This proves (3).

From (3) we may assume that u_4, u_5, v_1 all belong to the same component of $H \setminus v_3$.

(4) *We may assume that $v_2' = v_2$.*

For let S be the union of R_4, Q_4, the subpath of H between u_4 and u_5, Q_5, and R_5. If $v_2' \neq v_2$ then there is a triad with feet v_1, v_2, v_3 disjoint from S in the union of the subpath of H between v_2 and v_3, P_2, the subpath of J between u_1 and u_2, P_1 and R_1, and so $\mathcal{P}$ is strongly feasible as required. This proves (4).

From (3) and (4) we deduce that the hypotheses of (8.3) hold (with v_1, v_3 exchanged), taking C to be the union of the subpath of H between v_1' and $v_2' = v_2$, P_2, the subpath of J between u_1 and u_2, and P_1. The result follows from (8.3). ∎

It is convenient to prove a slight strengthening of (8.4). Let $v_1, \dots, v_5 \in V(G)$ be distinct. A *bat* in G on $\{v_1, \dots, v_5\}$ with *feet* v_4, v_5 is a set of six disjoint fragments of G, which can be numbered $X_1, \dots, X_6$ so that X_4X_5 are adjacent, and for $1 \leq i \leq 5, v_i \in X_i$ and X_iX_6 are adjacent.

(8.5) *Under the hypothesis of* (8.4), *there is a bat in G on $\{v_1, \dots, v_5\}$ with feet v_4, v_5.*

Proof. By (8.4), $\{v_1, v_2, v_3\}, \{v_4, v_5\}$ is strongly feasible, and so there is a path P between v_4 and v_5 and a fragment X_6 of G such that $X_6, V(P), \{v_1, v_2, v_3\}$ are mutually disjoint and v_1, v_2, v_3 all have neighbours in X_6. Choose X_6 maximal, and let N be the set of all $v \in V(G) - X_6$ with a neighbour in X_6. Then

$(V(G) - X_6, N \cup X_6)$ is a separation of G. Since $v_1, \ldots, v_5 \in V(G) - X_6 \neq V(G)$, it follows that this separation has order ≥ 5 (by(8.4)(iii)), and so $|N| \geq 5$. But $N - V(P) \subseteq \{v_1, v_2, v_3\}$ by the maximality of X_6, and so $|N \cap V(P)| \geq 2$. Choose an edge e of P so that N meets both components of $P \setminus e$; and let these components have vertex sets X_4, X_5 where $v_i \in X_i$ $(i = 4, 5)$. Let $X_i = \{v_i\}$ $(i = 1, 2, 3)$; then $\{X_1, \ldots, X_6\}$ is the desired bat. ∎

Let $v_1, \ldots, v_6 \in V(G)$ be distinct. We denote by $\mathcal{P}_1, \ldots, \mathcal{P}_{12}$ the following partitions:

$$\mathcal{P}_1 : \{v_1\}, \{v_3, v_5\}, \{v_2, v_4, v_6\}$$
$$\mathcal{P}_2 : \{v_2\}, \{v_3, v_5\}, \{v_1, v_4, v_6\}$$
$$\mathcal{P}_3 : \{v_3\}, \{v_2, v_5\}, \{v_1, v_4, v_6\}$$
$$\mathcal{P}_4 : \{v_4\}, \{v_2, v_5\}, \{v_1, v_3, v_6\}$$
$$\mathcal{P}_5 : \{v_5\}, \{v_2, v_4\}, \{v_1, v_3, v_6\}$$
$$\mathcal{P}_6 : \{v_6\}, \{v_2, v_4\}, \{v_1, v_3, v_5\}$$
$$\mathcal{P}_7 : \{v_1, v_3\}, \{v_2, v_5\}, \{v_4, v_6\}$$
$$\mathcal{P}_8 : \{v_1, v_4\}, \{v_2, v_5\}, \{v_3, v_6\}$$
$$\mathcal{P}_9 : \{v_1, v_4\}, \{v_2, v_6\}, \{v_3, v_5\}$$
$$\mathcal{P}_{10} : \{v_1, v_5\}, \{v_2, v_4\}, \{v_3, v_6\}$$
$$\mathcal{P}_{11} : \{v_1, v_6\}, \{v_2, v_4\}, \{v_3, v_5\}$$
$$\mathcal{P}_{12} : \{v_1, v_3, v_5\}, \{v_2, v_4, v_6\}.$$

(8.6) *Let G be a non-apex Hadwiger graph, and let (A, B) be a 6-separation of G with $|A - B|, |B - A| \geq 2$, chosen with A minimal. Let $G \mid A \cap B$ be a path with vertices $v_1, \ldots, v_6$ in order. Then $\mathcal{P}_1, \ldots, \mathcal{P}_{12}$ are infeasible in $G \mid A$.*

Proof. We show first that

(1) *There is an octopus $\{Y_1, \ldots, Y_8\}$ in $G \mid B$ on $\{v_1, \ldots, v_6\}$ with base v_1, v_4, v_6 such that Y_2Y_7, Y_3Y_7, Y_5Y_8 are adjacent.*

For by (6.9) there is an octopus $\{Y_1, \ldots, Y_8\}$ in $G \mid B$ on $\{v_1, \ldots, v_6\}$ with base v_1, v_4, v_6. We may assume by exchanging Y_7 and Y_8 that Y_2Y_7 are adjacent. If Y_3Y_8 and Y_5Y_7 are adjacent, let $\{C_1, C_3, C_4, C_5\}$ be (by (6.6)) a 4-cluster in $G \mid (A - \{v_2, v_6\})$ with $v_i \in C_i$ $(i = 1, 3, 4, 5)$; then

$$\{Y_1 \cup C_1, Y_2 \cup Y_7, Y_3 \cup C_3, Y_4 \cup C_4, Y_5 \cup C_5, Y_6 \cup Y_8\}$$

is a 6-cluster, a contradiction. If Y_3Y_8 and Y_5Y_8 are adjacent, let $\{C_1, C_2, C_5, C_6\}$ be a 4-cluster in $G \mid (A - \{v_3, v_4\})$ with $v_i \in C_i$ $(i = 1, 2, 5, 6)$; then

$$\{Y_1 \cup C_1, Y_2 \cup C_2, Y_3 \cup Y_8, Y_4 \cup Y_7, Y_5 \cup C_5, Y_6 \cup C_6\}$$

is a 6-cluster, a contradiction. Thus Y_3Y_8 are not adjacent, and so Y_3Y_7 are adjacent. If Y_5Y_7 are adjacent, let $\{C_1, C_3, C_5, C_6\}$ be a 4-cluster in $G \mid (A - \{v_2, v_4\})$ with $v_i \in C_i$ $(i = 1, 3, 5, 6)$; then

$$\{Y_1 \cup C_1, Y_2 \cup Y_7, Y_3 \cup C_3, Y_4 \cup Y_8, Y_5 \cup C_5, Y_6 \cup C_6\}$$

is a 6-cluster, a contradiction. Thus Y_5Y_8 are adjacent. This proves (1).

(2) *$\mathcal{P}_1,\ldots,\mathcal{P}_6$ are infeasible in $G \mid A$.*

For let $Y_1,\ldots,Y_8$ be as in (1). Now $\{v_2,v_4,v_6\}$ is stable. Moreover, if (A',B') is a separation of $G \mid (A-\{v_1\})$ with $\{v_2,\ldots,v_6\} \subseteq A'$ and $B'-A' \neq \emptyset$, then $(A' \cup B, B' \cup \{v_1\})$ is a separation of G of order $|A' \cap B'|+1$, and so $|A' \cap B'| \geq 5$, and by the minimality of A, either $B' \cup \{v_1\} = A$ (that is, $|A'|=5$) or $|B'-A'| = 1$. Hence the hypotheses of (8.5) applied to $G \mid (A-\{v_1\})$ are satisfied. Suppose that $\mathcal{P}_1$ is feasible. By (6.6) and (8.5) applied to $G \mid (A-\{v_1\})$, there is a bat $\{X_2,X_3,X_4,X_5,X_6,X_7\}$ in $G \mid (A-\{v_1\})$ on $\{v_2,\ldots,v_6\}$ with $v_i \in X_i$ $(2 \leq i \leq 6)$ and with feet v_3,v_5. Then

$$\{Y_1 \cup Y_2 \cup X_2 \cup Y_8, X_3 \cup Y_3, X_4 \cup Y_4, X_5 \cup Y_5, X_6 \cup Y_6 \cup Y_7, X_7\}$$

is a 6-cluster, a contradiction. Thus $\mathcal{P}_1$ is infeasible, and so is $\mathcal{P}_6$, by symmetry.

Suppose that $\mathcal{P}_2$ is feasible. By (6.6) and (8.5) there is a bat $\{X_1, X_3, X_4, X_5, X_6, X_7\}$ in $G \mid (A-\{v_2\})$ on $\{v_1,v_3,v_4,v_5,v_6\}$ with $v_i \in X_i$ $(i=1,3,4,5,6)$ and with feet v_3,v_5. Then

$$\{X_1 \cup Y_1 \cup Y_2 \cup Y_8, X_3 \cup Y_3, X_4 \cup Y_4, X_5 \cup Y_5, X_6 \cup Y_6 \cup Y_7, X_7\}$$

is a 6-cluster, a contradiction. Hence $\mathcal{P}_2$ is infeasible, and by symmetry so is $\mathcal{P}_5$.

Finally, suppose that $\mathcal{P}_3$ is feasible. By (6.6) and (8.5) there is a bat $\{X_1,X_2,X_4,X_5,X_6,X_7\}$ in $G \mid (A-\{v_3\})$ on $\{v_1,v_2,v_4,v_5,v_6\}$ with $v_i \in X_i$ $(i = 1,2,4,5,6)$ and with feet v_2,v_5. Then

$$\{X_1 \cup Y_1 \cup Y_8, X_2 \cup Y_2, Y_3 \cup X_4 \cup Y_4, X_5 \cup Y_5, X_6 \cup Y_6 \cup Y_7, X_7\}$$

is a 6-cluster, a contradiction. Hence $\mathcal{P}_3$ and similarly $\mathcal{P}_4$ are infeasible. This proves (2).

(3) *$\mathcal{P}_7$ is infeasible in $G \mid A$.*

For suppose that P_1,P_2,P_3 are three disjoint paths of $G \mid A$, where P_1 has ends v_1v_3, P_2 has ends v_4v_6, and P_3 has ends v_2v_5. We claim that there are two disjoint paths of $G \mid (A-\{v_1,v_3,v_4,v_6\})$ from $(V(P_1)-\{v_1,v_3\}) \cup \{v_2\}$ to $(V(P_2)-\{v_4,v_6\}) \cup \{v_5\}$. For if not, there is a (≤ 1)-separation (X,Y) of $G \mid (A-\{v_1,v_3,v_4,v_6\})$ with $V(P_1)-\{v_1,v_3\} \subseteq X$, $v_2 \in X$, $V(P_2)-\{v_4,v_6\} \subseteq Y$ and $v_5 \in Y$. Then $(X \cup B, Y \cup \{v_1,v_3,v_4,v_6\})$ has order ≤ 6, and so from the minimality of A, either $X \cup B = B$ or $|V(G)-(X \cup B)| \leq 1$. The first is impossible since $V(P_1) \subseteq X$ and P_1 has an internal vertex (for v_1v_3 are not adjacent). Thus the second holds, and so $|Y-(X \cup B)| \leq 1$. Similarly $|X-(Y \cup B)| \leq 1$; but also $|X \cap Y - B| \leq 1$, and so $|V(G)-B| \leq 3$, contrary to (6.3). This proves our claim that there exist two disjoint paths Q_1,Q_2 of $G \mid (A-\{v_1,v_3,v_4,v_6\})$ from $(V(P_1)-\{v_1,v_3\}) \cup \{v_2\}$ to $(V(P_2)-\{v_4,v_6\}) \cup \{v_5\}$; and because of the existence of P_3, we may choose Q_1,Q_2 so that v_2 is an end of one of them, and so is v_5, and Q_1,Q_2 have no vertex in $V(P_1)$ except for p_1, say, and no vertex in $V(P_2)$ except for p_2, say. Now if one of Q_1,Q_2 has ends v_2v_5 and the other has ends p_1p_2, then $\mathcal{P}_4$ is feasible in $G \mid A$, contrary to (2); while if one of Q_1,Q_2 has ends v_2p_2 and the other has ends p_1v_5 then $\mathcal{P}_1$ is feasible contrary to (2). This proves (3).

(4) *$\mathcal{P}_8, \mathcal{P}_9, \mathcal{P}_{10}$ are infeasible in $G \mid A$.*

Now $\mathcal{P}_8$ is infeasible by (7.6) and (6.6), taking $Z_1 = \{v_2\}, Z_2 = \{v_5\}$, $Z_3 = \{v_1, v_4\}, Z_4 = \{v_3, v_6\}$. $\mathcal{P}_9$ is infeasible by (7.6) and (6.6), taking $Z_1 = \{v_3\}, Z_2 = \{v_5\}$, $Z_3 = \{v_1, v_4\}, Z_4 = \{v_2, v_6\}$. $\mathcal{P}_{10}$ is infeasible by symmetry.

(5) *$\mathcal{P}_{11}$ is infeasible in $G \mid A$.*

For suppose that there are disjoint paths P, Q and R of $G \mid A$ with ends v_1v_6, v_2v_4 and v_3v_5 respectively. Let S be a minimal path of $G \mid (A - \{v_2, v_5\})$ between $V(P)$ and $V(Q \cup R)$; let S have ends $p \in V(P)$ and $q \in V(Q)$ say. (This exists since there is a path of $G \mid A$ between v_1 and v_3 with no vertex in $\{v_2, v_5\}$, because any component of $G \setminus B$ contains neighbours of all of $v_1, \ldots, v_6$.) Since $q \neq v_2$ it follows that $\mathcal{P}_2$ is feasible, a contradiction. This proves (5).

(6) *$\mathcal{P}_{12}$ is infeasible in $G \mid A$.*

For suppose it is feasible. Since $\mathcal{P}_1$ and $\mathcal{P}_6$ are infeasible, it follows that there are four paths $P_{13}, P_{15}, P_{26}, P_{46}$ of $G \mid A$, mutually disjoint except for their ends, where each P_{ij} has ends v_iv_j. But by (1), $\{v_3\}, \{v_1, v_5\}$, $\{v_2, v_4, v_6\}$ is feasible in $G \mid B$. This contradicts (7.6) with $Z_1 = \{v_3\}$, $Z_2 = \{v_1, v_5\}$, $Z_3 = \{v_2, v_4, v_6\}$.

The result follows. ∎

In addition to (8.6) we need to prove that one further structure does not appear in $G \mid A$. Let $v_1, \ldots, v_5$ be distinct vertices of a graph G. A *turkey* in G on $(v_1, \ldots, v_5)$ (note that here $v_1, \ldots, v_5$ are ordered, unlike the octopus and bat) is a set $\{X_0, X_1, X_2, X_3, X_5\}$ of disjoint fragments of G, such that

(i) $v_1, v_4 \in X_1$, and $v_i \in X_i$ for $i = 2, 3, 5$, and

(ii) $X_0X_1, X_0X_2, X_0X_3, X_2X_5$ and X_3X_5 are adjacent.

If $(v_1, \ldots, v_6)$ is a 6-term sequence, and $1 \le k \le 6$, the 5-term sequence obtained by omitting the kth term v_k of $(v_1, \ldots, v_6)$ is denoted by $(v_1, \ldots, \hat{v}_k, \ldots, v_6)$.

(8.7) *Let $G, A, B, v_1, \ldots, v_6$ be as in (8.6) Let $1 \le k \le 6$; then there is no turkey in $G \mid A$ on $(v_1, \ldots, \hat{v}_k, \ldots, v_6)$.*

Proof. Suppose that for some k there exists such a turkey. Let $(v_1, \ldots, \hat{v}_k, \ldots, v_6) = (a_1, \ldots, a_5)$. Then there are five paths R_i of $G \mid A$ with ends a_i, b_i $(1 \le i \le 5)$, mutually disjoint except that $b_1 = b_4$; and two paths Q_i of $G \mid A$ between b_i and b_5 $(i = 2, 3)$; a vertex $c \in A - V(R_1 \cup \ldots \cup R_5 \cup Q_1 \cup Q_2)$, and three paths P_i from c to b_i $(i = 1, 2, 3)$, so that all these paths are disjoint except for their ends. (Note that it is possible that $a_i = b_i$ for some values of i.) See figure 3.

Let $H = R_1 \cup \ldots \cup R_5 \cup Q_1 \cup Q_2 \cup P_1 \cup P_2 \cup P_3$; we call H a *skeleton*.

Now since G is 6-connected, there are six paths $L_1, \ldots, L_6$ of $G \mid A$ between c and $v_1, \ldots, v_6$ respectively, mutually disjoint except for c. Choose k, H and $L_1, \ldots, L_6$ so that

(1) *$H \cup L_1 \cup \ldots \cup L_6$ is minimal.*

Let R be the minimal subpath of L_k between v_k and $V(H)$, and let R have ends v_k, h. Thus, if $v_k \in V(H)$ then $h = v_k$.

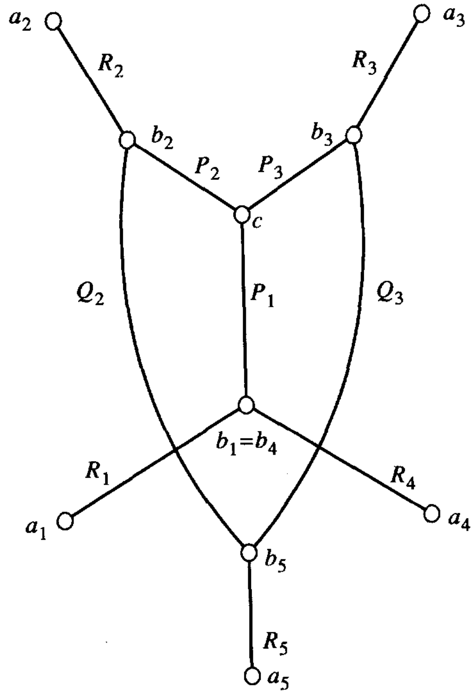

Fig. 3. A turkey skeleton on $(a_1,\ldots,a_5)$

(2) *For* $1 \leq i \leq 5$, *if* a_i *and* v_k *are consecutive in the sequence* $(v_1,\ldots,v_6)$, *then* $h \notin V(R_i)$.

For suppose that a_i and v_k are consecutive in $(v_1,\ldots,v_6)$ and $h \in V(R_i)$. We obtain a new skeleton H' by replacing the subpath R' of R_i between a_i and h by R. But

$$H' \cup L_1 \cup \ldots \cup L_6 \subseteq H \cup L_1 \cup \ldots \cup L_6,$$

and so by (1) equality holds; and hence $R' \subseteq L_1 \cup \ldots \cup L_6$, which is impossible since $R \subseteq L_k$, $R' \subseteq L_i, R \cap R'$ is non-null (since $h \in V(R \cap R')$), and $L_1,\ldots,L_6$ are disjoint except for c. This proves (2).

(3) *For* $1 \leq i \leq 5, h \notin V(R_i)$.

Suppose that $h \in V(R_i)$. By (2), a_i and v_k are not consecutive in $(v_1,\ldots,v_6)$. There are several cases.

Suppose that $k=1$, and hence $i \geq 2$ and $a_i = v_{i+1}$. Then $i \neq 2$ (for otherwise $\mathcal{P}_4$ is feasible, contrary to (8.6); in the remaining cases we abbreviate this to "($\mathcal{P}_4$)"), and $i \neq 3$ ($\mathcal{P}_3$). Now $h \neq b_4$ since $h \notin V(R_1)$, and so $i \neq 4$ ($\mathcal{P}_{10}$), and $i \neq 5$ ($\mathcal{P}_4$). Hence $k \neq 1$.

Suppose that $k=2$, and hence $i \neq 1,2$. Then $i \neq 3$ ($\mathcal{P}_6$), $i \neq 4$ ($\mathcal{P}_3$) since $h \neq b_4$, and $i \neq 5$ ($\mathcal{P}_{12}$). Thus, $k \neq 2$.

Suppose that $k=3$. Then $i \neq 1$ or 4 ($\mathcal{P}_6$), and $i \neq 5$ ($\mathcal{P}_{10}$). Hence $k \neq 3$.

Suppose that $k = 4$. Then $i \neq 1$ ($\mathcal{P}_8$) since $h \neq b_1$, and $i \neq 2$ ($\mathcal{P}_6$) and $i \neq 5$ ($\mathcal{P}_{12}$). Thus, $k \neq 4$.

Suppose that $k = 5$. Then $i \neq 1$ ($\mathcal{P}_{10}$) since $h \neq b_1$, and $i \neq 2$ ($\mathcal{P}_8$), and $i \neq 3$ ($\mathcal{P}_9$). Hence, $k \neq 5$.

Suppose that $k = 6$. Then $i \neq 1$ or 4 ($\mathcal{P}_3$), $i \neq 2$ ($\mathcal{P}_9$) and $i \neq 3$ ($\mathcal{P}_8$). Hence, $k \neq 6$.

In each case we therefore obtain a contradiction, and (3) follows.

(4) $h \notin V(Q_2 \cup Q_3)$.

For suppose first that $h \in V(Q_2)$. Since $h \notin V(R_2 \cup R_5)$ by (3), $h \neq b_2$ and $h \neq b_5$. Then $k \neq 1$ ($\mathcal{P}_4$), $k \neq 2$ ($\mathcal{P}_{12}$), $k \neq 3$ ($\mathcal{P}_{10}$), $k \neq 4$ ($\mathcal{P}_6$), $k \neq 5$ ($\mathcal{P}_8$), and $k \neq 6$ ($\mathcal{P}_9$), a contradiction. Hence $h \notin V(Q_2)$. Suppose now that $h \in V(Q_3)$. By (3), $h \neq b_3, b_5$. Hence $k \neq 1$ ($\mathcal{P}_3$), $k \neq 2$ ($\mathcal{P}_6$), $k \neq 3$ ($\mathcal{P}_{10}$), $k \neq 4$ ($\mathcal{P}_{12}$), $k \neq 5$ ($\mathcal{P}_9$), and $k \neq 6$ ($\mathcal{P}_8$). This proves (4).

From (3) and (4), we deduce that $h \in V(P_1 \cup P_2 \cup P_3)$. By (3), $h \neq b_1, b_2, b_3$. Hence $k \neq 1$ (P_3), $k \neq 2$ ($\mathcal{P}_{10}$), $k \neq 3$ ($\mathcal{P}_{12}$), $k \neq 4$ ($\mathcal{P}_{10}$), $k \neq 5$ ($\mathcal{P}_8$), and $k \neq 6$ ($\mathcal{P}_3$). This is a contradiction, and so there is no such turkey, as required. ∎

9. Chasing a turkey

Let $a_1, \ldots, a_5$ be distinct vertices of a graph G, fixed throughout this section. $\mathcal{P}$ denotes the partition $\{a_1, a_3, a_5\}, \{a_2, a_4\}$, and we assume the following three hypotheses:

(9.1) (Hypothesis) *$\mathcal{P}$ is infeasible in G.*

(9.2) (Hypothesis) *There is no turkey in G on $(a_1, \ldots, a_5)$ or on $(a_5, \ldots, a_1)$.*

(9.3) (Hypothesis) *G is simple, and there is no separation (X, Y) of G of order ≤ 3 with $a_1, \ldots, a_5 \in X$ and $|V(G) - X| \geq 2$, and none of order ≤ 2 with $a_1, \ldots, a_5 \in X \neq V(G)$.*

A *frame* on $(a_1, \ldots, a_5)$ (see figure 4) is a subgraph H of G with $a_1, \ldots, a_5 \in V(H)$, consisting of the union of:

(i) five paths P_i with ends $u_i a_i$ $(1 \leq i \leq 5)$, mutually vertex-disjoint, where $u_1 \neq a_1$ and $u_5 \neq a_5$

(ii) two disjoint paths R_{17}, R_{56}, with ends $u_1 u_7$ and $u_5 u_6$ respectively, meeting $V(P_1 \cup \ldots \cup P_5)$ in $\{u_1\}$ and $\{u_5\}$ respectively

(iii) six paths $Q_{12}, Q_{26}, Q_{36}, Q_{37}, Q_{47}, Q_{45}$, mutually disjoint except for their ends, where each Q_{ij} has ends $u_i u_j$, and each Q_{ij} is disjoint from $P_1, \ldots, P_5$ and R_{17}, R_{56} except for its ends.

This definition implies that the paths $P_1, P_5, Q_{12}, Q_{26}, Q_{36}, Q_{37}, Q_{47}, Q_{45}$ each have at least one edge, but the paths $P_2, P_3, P_4, R_{17}, R_{56}$ may have no edges. In particular, we permit $u_5 = u_6$ and $u_1 = u_7$. We call $P_1, P_2, P_3, P_4, P_5, Q_{12}, Q_{26}, Q_{36}, Q_{37}, Q_{47}, Q_{45}$ the *sides* of the frame, and denote their union by $I(H)$. We define the *cost* of H to be $|E(R_{17})| + |E(R_{56})|$. A frame in G on $(a_1, \ldots, a_5)$ is *minimal* if its cost is minimum over all frames in G on $(a_1, \ldots, a_5)$.

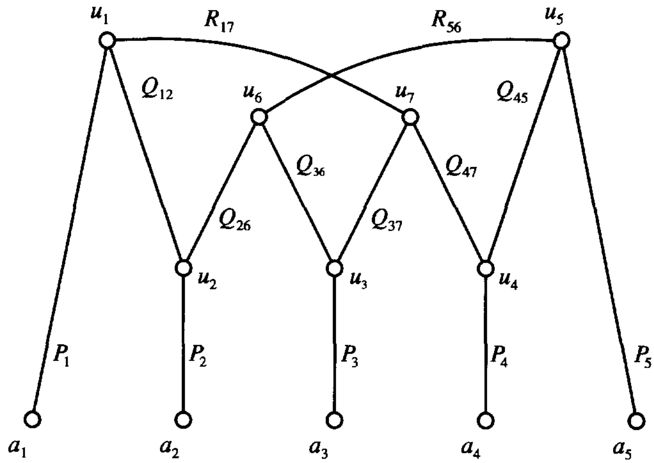

Fig. 4. A frame

The objective of this section is to analyze the structure of G implied by (9.1)–(9.3), assuming there is a frame. Roughly, we shall show that G can be drawn in a disc with $a_1,\ldots,a_5$ on the boundary in order, except for one part of G (associated with $R_{17} \cup R_{56}$) which is separated from the remainder of G by a (≤ 4)-separation.

If H is a subgraph of G, let us say an *H-path* in G is a path of G with distinct ends both in $V(H)$, and with no other vertex or edge in H. We begin with the following.

(9.4) *Assuming* (9.1)–(9.3), *let H be a minimal frame on $(a_1,\ldots,a_5)$ with notation as above. There is no path in $G \setminus \{u_1,u_5,u_6,u_7\}$ between $V(R_{17} \cup R_{56}) - \{u_1,u_5,u_6,u_7\}$ and $V(I(H)) - \{u_1,u_5,u_6,u_7\}$.*

Proof. Suppose there is such a path; then there is an H-path P in G with ends $x \in V(R_{17} \cup R_{56})$ and $y \in V(I(H))$, with $x,y \notin \{u_1,u_5,u_6,u_7\}$. From the symmetry we may assume that $x \in V(R_{17})$.

(1) $y \notin V(P_1 \cup Q_{12} \cup P_2 \cup Q_{26})$.

For if $y \in V(P_2 \cup Q_{26})$ we replace Q_{12} by P to obtain a new frame with smaller cost, a contradiction. If $y \in V(P_1)$ or $y \in V(Q_{12})$ we replace the subpath of P_1 or Q_{12} respectively between y and u_1 by P, again obtaining a frame with smaller cost. This proves (1).

(2) $y \notin V(Q_{36} \cup P_3 \cup Q_{37})$ *and* $y \notin V(Q_{47} \cup P_4 \cup Q_{45})$.

For if $y \in V(Q_{36} \cup P_3 \cup Q_{37})$ we replace Q_{37} (or a part of it) by P, obtaining a frame with smaller cost. If $y \in V(Q_{47} \cup P_4 \cup Q_{45})$, we similarly replace Q_{47} (or a part of it) by P.

From (1) and (2) we deduce that $y \in V(P_5) - \{u_5\}$. But then $\mathscr{P}$ is feasible, contrary to (9.1). ∎

Given a frame H and P_i's and Q_{ij}'s as before, we define

$$\begin{aligned}
R_1 &= P_1 \cup Q_{12} \cup P_2 \\
R_2 &= Q_{12} \cup Q_{26} \\
R_3 &= P_2 \cup Q_{26} \cup Q_{36} \cup P_3 \\
R_4 &= Q_{36} \cup Q_{37} \\
R_5 &= P_3 \cup Q_{37} \cup Q_{47} \cup P_4 \\
R_6 &= Q_{47} \cup Q_{45} \\
R_7 &= P_4 \cup Q_{45} \cup P_5.
\end{aligned}$$

Then $R_1,\dots,R_7$ are all paths of $I(H)$.

(9.5) *Assuming* (9.1)–(9.3), *let H be a minimal frame on $(a_1,\dots,a_5)$. Let P be an H-path of G with ends $x,y \in V(H)$. Then either $x,y \in V(R_i)$ for some i $(1 \le i \le 7)$, or $x,y \in V(R_{17} \cup R_{56})$.*

Proof. First, suppose that $x \in V(P_1) - \{u_1\}$. We must show that $y \in V(R_1)$. Now since $\mathcal{P}$ is not feasible, a_1,a_3,a_5 do not belong to the same component of $G \setminus V(P_2 \cup Q_{12} \cup R_{17} \cup Q_{47} \cup P_4)$, and so

$$y \notin V(Q_{26} \cup Q_{36} \cup P_3 \cup Q_{37} \cup R_{56} \cup Q_{45} \cup P_5) - \{u_2,u_4,u_7\}.$$

We deduce by (9.4) that $y \in V(P_4 \cup Q_{47}) \cup V(R_1)$. Now if $y \in V(P_4 \cup Q_{47}) - \{u_7\}$, then taking

$$\begin{aligned}
X_1 &= V(P_1 \cup P \cup P_4 \cup Q_{47}) - \{u_1,u_7\} \\
X_2 &= V(P_2) \\
X_3 &= V(P_3) \\
X_5 &= V(P_5 \cup R_{56} \cup Q_{26} \cup Q_{36}) - \{u_2,u_3\} \\
X_0 &= V(Q_{12} \cup R_{17} \cup Q_{37}) - \{u_2,u_3\}
\end{aligned}$$

defines a turkey on $(a_1,\dots,a_5)$ contrary to (9.2). If $y = u_7 \neq u_1$, then replacing the subpath of P_1 between x and u_1 by P yields a frame with smaller cost, a contradiction. It follows that if $y \in V(P_4 \cup Q_{47})$ then $y = u_7 = u_1$ and so $y \in V(R_1)$. Hence the result holds if $x \in V(P_1) - \{u_1\}$. We may therefore assume by symmetry that

(1) *$x,y \notin V(P_1) - \{u_1\}$ and $x,y \notin V(P_5) - \{u_5\}$.*

Next, we claim we may assume that

(2) *$x,y \neq u_1,u_5$.*

For if not then by symmetry we may assume that $x = u_1$. If

$$y \in V(Q_{36} \cup P_3 \cup Q_{37} \cup Q_{47} \cup P_4 \cup Q_{45}) - \{u_5,u_6,u_7\}$$

then from the minimality of H, it follows (by replacing all or a part of one of Q_{37},Q_{47}) that $u_1 = u_7$, and hence $x,y \in V(R_i)$ for $i = 4,5$ or 6 and the result is true. By (1), $y \notin V(P_5) - \{u_5\}$. Thus either $y = u_5$ or u_7 (in which case $x, y \in V(R_{17} \cup R_{56})$) or $y \in V(R_1) \cup V(R_2)$, and then $x, y \in V(R_i)$ for $i = 1$ or 2. This proves (2).

Next, we claim we may assume that

(3) $x, y \notin V(Q_{12}) - \{u_2\}$ *and* $x, y \notin V(Q_{45}) - \{u_4\}$.

For otherwise, by (2) we may assume that $x \in V(Q_{12}) - \{u_1, u_2\}$. Now by (9.1), $y \notin V(P_4 \cup Q_{47} \cup Q_{45}) - \{u_5, u_7\}$, since otherwise a_1, a_3, a_5 belong to the same component of

$$G \setminus (V(P_2 \cup Q_{12} \cup P \cup P_4 \cup Q_{47} \cup Q_{45}) - \{u_1, u_5, u_7\}).$$

By the minimality of $H, y \neq u_7$ (for if $y = u_7$ then $u_7 \neq u_1$ by (2), and so replacing part of Q_{12} by P produces a frame with smaller cost). If $y \in V(Q_{36} \cup P_3 \cup Q_{37}) - \{u_6, u_7\}$, then taking

$$\begin{aligned}
X_1 &= V(P_1 \cup R_{17} \cup Q_{47} \cup P_4)\\
X_2 &= V(P_2 \cup Q_{26}) - \{u_6\}\\
X_3 &= V(Q_{36} \cup P_3 \cup Q_{37}) - \{u_6, u_7\}\\
X_5 &= V(P_5 \cup R_{56})\\
X_0 &= V(Q_{12} \cup P) - \{u_1, u_2, y\}
\end{aligned}$$

defines a turkey on $(a_1, \ldots, a_5)$ contrary to (9.2). By (1) and (2), $y \notin V(P_5)$; and so $y \in V(R_1) \cup V(R_2)$ as required. This proves (3).

Next, we claim we may assume that

(4) $x, y \notin V(P_2 \cup Q_{26}) - \{u_6\}$ *and* $x, y \notin V(P_4 \cup Q_{47}) - \{u_7\}$.

For suppose that $x \in V(P_2 \cup Q_{26}) - \{u_6\}$ say. By (1)–(3),

$$y \in V(P_4 \cup Q_{47} \cup Q_{37} \cup R_3).$$

Now $y \notin V(P_4 \cup Q_{47}) - \{u_7\}$ since $\mathcal{P}$ is not feasible. Also, $y \neq u_7$, for if $y = u_7$ then $u_7 \neq u_1$ by (2), and replacing Q_{12} by P contradicts the minimality of H. If $y \in V(Q_{37}) - \{u_3, u_7\}$, then taking

$$\begin{aligned}
X_1 &= V(P_1 \cup R_{17} \cup Q_{47} \cup P_4)\\
X_2 &= V(P_2 \cup Q_{26} \cup P) - \{u_6, y\}\\
X_3 &= V(Q_{36} \cup P_3) - \{u_6\}\\
X_5 &= V(P_5 \cup R_{56})\\
X_0 &= V(Q_{37}) - \{u_3, u_7\}
\end{aligned}$$

defines a turkey on $(a_1, \ldots, a_5)$, a contradiction. Hence $y \in V(R_3)$ as required. This proves (4).

From (1)–(4), $x, y \in V(P_3 \cup Q_{36} \cup Q_{37})$, and so $x, y \in V(R_i)$ for $i = 3$, 4 or 5, as required. ∎

If $C \subseteq V(G)$, we denote by $N(C)$ the set of all $v \in V(G) - C$ with a neighbour in C. We recall that if H is a subgraph of G, an H-flap is the vertex set of a component of $G \setminus V(H)$.

(9.6) *Assuming* (9.1)–(9.3), *let* H *be a minimal frame on* $(a_1,\dots,a_5)$. *If* C *is an* H*-flap and* $N(C)$ *meets each of* $V(Q_{12})-\{u_2\}$, $V(P_2)-\{u_2\}$, $V(Q_{26})-\{u_2\}$, *then* $N(C)\subseteq V(Q_{12}\cup P_2\cup Q_{26})$. *An analogous result holds for* Q_{36},P_3,Q_{37}.

Proof. Since $N(C)\cap(V(P_2)-\{u_2\})\neq\emptyset$, it follows from (9.5) that $N(C)\subseteq V(R_1\cup R_3)$. Similarly $N(C)\subseteq V(R_1\cup R_2)$, and $N(C)\subseteq V(R_2\cup R_3)$. Hence

$$N(C)\subseteq V(R_1\cup R_3)\cap V(R_1\cup R_2)\cap V(R_2\cup R_3)=V(Q_{12}\cup P_2\cup Q_{26}).$$

The proof is analogous for Q_{36},P_3,Q_{37}. ∎

(9.7) *Assuming* (9.1)–(9.3), *let* H *be a minimal frame on* $(a_1,\dots,a_5)$. *There is no triad in* G *with feet in* $V(Q_{12})-\{u_2\},V(P_2)-\{u_2\}$ *and* $V(Q_{26})-\{u_2\}$ *respectively and with no other vertex in* $V(H)$.

Proof. Suppose that T is such a triad with feet x_1,x_2,x_3 say, where $x_1\in V(Q_{12})-\{u_2\}$, $x_2\in V(P_2)-\{u_2\}$ and $x_3\in V(Q_{26})-\{u_2\}$ respectively. Then $T\cup P_2\cup Q_{12}\cup Q_{26}$ is a tripod with feet u_1,a_2,u_6 and with no other vertex in

$$Z=V(H)-(V(P_2\cup Q_{12}\cup Q_{26})-\{u_1,a_2,u_6\}).$$

By (3.3) and (9.3), we may assume (by the symmetry between the two triads in the tripod) that there is a path of G from $V(T)-\{x_1,x_2,x_3\}$ to Z disjoint from $P_2\cup Q_{12}\cup Q_{26}$, contrary to (9.6). ∎

A virtually identical proof yields:

(9.8) *Assuming* (9.1)–(9.3), *let* H *be a minimal frame on* $(a_1,\dots,a_5)$. *There is no triad in* G *with feet in* $V(Q_{36})-\{u_3\}$, $V(P_3)-\{u_3\}$ *and* $V(Q_{37})-\{u_3\}$ *respectively and with no other vertex in* $V(H)$.

(9.9) *Assuming* (9.1)–(9.3), *let* H *be a minimal frame on* $(a_1,\dots,a_5)$. *For any* H*-flap* C, *either* $N(C)\subseteq V(R_{17}\cup R_{56})$ *or* $N(C)\subseteq V(R_i)$ *for some* i $(1\le i\le 7)$.

Proof. We may assume that there exists $x\in N(C)-V(R_{17}\cup R_{56})$. By (9.4), $N(C)\subseteq V(I(H))$. If x can be chosen with $x\in V(P_1)-\{u_1\}$ then by (9.5), $N(C)\subseteq V(R_1)$ as required, and we may therefore assume that $N(C)\cap V(P_1)\subseteq\{u_1\}$ and $N(C)\cap V(P_5)\subseteq\{u_5\}$ similarly. If $N(C)\subseteq V(Q_{36}\cup P_3\cup Q_{37})$, then by (9.8), $N(C)\subseteq V(R_i)$ for $i=3,4$ or 5, as required. From the symmetry, we may therefore assume that

$$N(C)\cap V(Q_{12}\cup P_2\cup Q_{26})\not\subseteq\{u_6\}.$$

From (9.5),

$$N(C)\subseteq V(Q_{12}\cup P_2\cup Q_{26}\cup Q_{36}\cup P_3).$$

We may assume that $N(C)\not\subseteq V(R_3)$, and so $N(C)\cap V(Q_{12})\not\subseteq\{u_2\}$. By (9.5), $N(C)\subseteq V(Q_{12}\cup P_2\cup Q_{26})$. Then the result follows from (9.7). ∎

(9.10) *Assuming* (9.1)–(9.3), *let* H *be a minimal frame on* $(a_1,\dots,a_5)$. *There do not exist disjoint* H*-paths* P, Q *with ends* p_1p_2 *and* q_1q_2 *respectively, such that* $p_1,q_1\in V(P_1)-\{u_1\}$, $p_2,q_2\in V(Q_{12})-\{u_1\}$, *and* a_1,p_1,q_1,p_2,q_2,a_2 *are in order on* R_1.

Proof. Suppose that such P and Q exist. Let P' be the subpath of P_1 between u_1 and p_1. Now $P_1 \cup Q_{12} \cup P \cup Q$ is a tripod with feet a_1, u_1, u_2 and with no other vertex in

$$Z = V(H) - (V(P_1 \cup Q_{12}) - \{a_1, u_1, u_2\}).$$

By (3.3) we may therefore assume that there is a path R from $a \in V(P' \cup Q) - \{u_1, p_1, q_2\}$ to $b \in Z$ with no vertex in $H \cup P \cup Q$ except a and b, and with $a_1, u_1, u_2 \neq a, b$. (We use here that every leg of the tripod "output" by (3.3) is a subpath of the corresponding leg of the "input" tripod, so that the leg incident with u_1 remains null, and we use the symmetry between q_1 and p_2.) By (9.5), $b \in V(R_1)$, and so

$$b \in V(R_1) \cap Z - \{a_1, u_1, u_2\} = V(P_2) - \{u_2\}.$$

But then $\mathcal{P}$ is feasible, a contradiction. ∎

(9.11) *Assuming* (9.1)–(9.3), *let* H *be a minimal frame on* $(a_1, \ldots, a_5)$. *There do not exist disjoint* H*-paths* P, Q *with ends* p_1p_2 *and* q_1q_2 *respectively, such that* $p_1, q_1 \in V(Q_{26}) - \{u_6\}$, $p_2, q_2 \in V(Q_{36}) - \{u_6\}$, *and* $a_2, p_1, q_1, p_2, q_2, a_3$ *are in order in* R_3.

Proof. Suppose such P, Q exist. Let P' be the subpath of Q_{26} between p_1 and u_6. As in (9.10), we may assume by (3.3) that there is a path R with ends $a \in V(P' \cup Q)$ and $b \in V(H) - V(Q_{26} \cup Q_{36})$, with no other vertex in $V(H \cup P \cup Q)$ and with no vertex in $\{p_1, q_2, u_6\}$. By (9.5), $b \in V(R_2 \cup R_3)$, and so either $b \in V(P_2) - \{u_2\}$, or $b \in V(P_3) - \{u_3\}$, or $b \in V(Q_{12}) - \{u_2\}$.

Suppose first that $b \in V(P_2) - \{u_2\}$. Let H' be obtained from H by deleting the edges and internal vertices of the subpath of P_2 between b and u_2, and adding R (if $a \in V(Q_{26})$) or adding R and the subpath of Q between a and q_1 (if $a \in V(Q)$). Then H' is a minimal frame on $(a_1, \ldots, a_5)$, and yet it does not satisfy (9.5), because of the H'-path P, a contradiction. Thus $b \notin V(P_2) - \{u_2\}$.

Now suppose that $b \in V(P_3) - \{u_3\}$. Then taking

$$\begin{aligned}
X_1 &= V(P_1 \cup R_{17} \cup Q_{47} \cup P_4)\\
X_2 &= V(P_2 \cup Q_{26}) - (V(P') - \{p_1\})\\
X_3 &= V(P_3) - \{u_3\}\\
X_5 &= V(P_5 \cup R_{56} \cup P' \cup Q \cup R) - \{p_1, b, q_2\}\\
X_0 &= V(P \cup Q_{36} \cup Q_{37}) - \{p_1, u_6, u_7\}
\end{aligned}$$

defines a turkey on $(a_1, \ldots, a_5)$ contrary to (9.2). Thus $b \notin V(P_3) - \{u_3\}$.

Consequently, $b \in V(Q_{12}) - \{u_2\}$. Let Q' be the subpath of Q_{36} between u_6 and p_2. Then taking

$$\begin{aligned}
X_1 &= V(P_1 \cup R_{17} \cup Q_{47} \cup P_4 \cup Q_{12}) - \{u_2\}\\
X_2 &= V(P_2) \cup (V(Q_{26}) - V(P')) \cup (V(P) - \{p_2\})\\
X_3 &= V(P_3) \cup (V(Q_{36}) - V(Q'))\\
X_5 &= (P_5 \cup R_{56} \cup Q')\\
X_0 &= V(R \cup P' \cup Q) - \{b, p_1, u_6, q_2\}
\end{aligned}$$

defines a turkey on $(a_1,\ldots,a_5)$ contrary to (9.2). (The reader may see what seems to be a simpler way to dispose of this case, but there is a difficulty with it if $b = u_1 = u_7$.) The result follows. ∎

We recall that $P_1,\ldots,P_5$ and the Q_{ij}'s are called the *sides* of H.

(9.12) *Assuming* (9.1)–(9.3), *let H be a minimal frame on $(a_1,\ldots,a_5)$. Let P, Q be disjoint H-paths of G, with ends p_1p_2 and q_1q_2 respectively, and let k with $1 \le k \le 7$ be such that p_1, q_1, p_2, q_2 all lie in R_k, in order. Then one of $\{p_1,p_2\}$, $\{q_1,q_2\}$ is a subset of the vertex set of some side of H.*

Proof. Suppose first that $k = 1$; then we may assume that a_1,p_1,q_1,p_2,q_2,a_2 are in order in R_1. If $q_2 \notin V(P_2) - \{u_2\}$, then the result holds by (9.10), and so we assume that $q_2 \in V(P_2) - \{u_2\}$. Suppose that $p_1 \notin V(P_1) - \{u_1\}$. Then either the result holds, or $p_1,q_1 \in V(Q_{12}) - \{u_2\}$ and $p_2,q_2 \in V(P_2) - \{u_2\}$; but in the latter case by replacing the subpath of P_2 between q_2 and u_2 by Q, we obtain a minimal frame in which (9.7) is not satisfied, a contradiction. Hence we may assume $p_1 \in V(P_1) - \{u_1\}$. Consequently, if the result does not hold, there are disjoint paths in $R_1 \setminus \{p_1,q_2\}$ with ends q_1,u_1 and p_2,u_2 respectively, and hence $\mathcal{P}$ is feasible, a contradiction. Thus if $k=1$ the result holds.

If $k=2$ the result holds, for otherwise there is a minimal frame violating (9.7), and similarly (using (9.8)) the result holds if $k = 4$. By the symmetry we may therefore assume that $k = 3$ and a_2,p_1,q_1,p_2,q_2,a_3 are in order in R_3. By (9.7), the result holds if $q_2 \in V(P_2 \cup Q_{26})$, and by (9.8) it holds if $p_1 \in V(Q_{36} \cup P_3)$. We assume therefore that $p_1 \in V(P_2 \cup Q_{26}) - \{u_6\}$, and $q_2 \in V(Q_{36} \cup P_3) - \{u_6\}$. If $p_2 \in V(P_2 \cup Q_{26})$, then $p_1 \in V(P_2) - \{u_2\}$ and $p_2 \in V(Q_{26}) - \{u_2\}$ (unless the result holds), and by replacing the subpath of Q_{26} between u_2 and p_2 by P we obtain a minimal frame violating (9.5). Thus we may assume that $p_2 \in V(Q_{36} \cup P_3) - \{u_6\}$. Similarly if $q_1 \in V(Q_{36} \cup P_3)$, we may assume that $q_2 \in V(P_3) - \{u_3\}$ and $q_1 \in V(Q_{36}) - \{u_3\}$, and then by replacing the subpath of Q_{36} between q_1 and u_3 by Q, we obtain a minimal frame violating (9.5). We therefore assume that $q_1 \in V(P_2 \cup Q_{26}) - \{u_6\}$. If $p_1 \in V(P_2) - \{u_2\}$ and $q_2 \in V(P_3) - \{u_3\}$ then $\mathcal{P}$ is feasible, a contradiction. Thus if $p_1 \in V(P_2) - \{u_2\}$, then $q_2 \in V(Q_{36})$; but then there is a turkey on $(a_1,\ldots,a_5)$, taking

$$
\begin{aligned}
X_1 &= V(P_1 \cup R_{17} \cup Q_{47} \cup P_4)\\
X_2 &= V(P')\\
X_3 &= V(P_3 \cup Q')\\
X_5 &= V(P_5 \cup R_{56} \cup Q_{36} \cup P) - (V(Q') \cup \{p_1\})\\
X_0 &= V(Q_{12} \cup P_2 \cup Q_{26} \cup Q) - (V(P') \cup \{u_1, u_6, q_2\})
\end{aligned}
$$

where P' is the subpath of P_2 between a_2 and p_1, and Q' is the subpath of Q_{36} between q_2 and u_3. Consequently, $p_1 \in V(Q_{26}) - \{u_6\}$.

By (9.11), $q_2 \notin V(Q_{36})$ (since $q_2 \neq u_6$), and so $q_2 \in V(P_3) - \{u_3\}$. But then there is a turkey on $(a_1, \ldots, a_5)$, taking

$$\begin{aligned}
X_1 &= V(P_1 \cup R_{17} \cup Q_{47} \cup P_4),\\
X_2 &= V(P_2 \cup P')\\
X_3 &= V(Q')\\
X_5 &= V(P_5 \cup R_{56} \cup Q_{26} \cup Q) - (V(P') \cup \{q_2\})\\
X_0 &= V(P_3 \cup Q_{36} \cup Q_{37} \cup P) - (V(Q') \cup \{p_1, u_6, u_7\})
\end{aligned}$$

where P' is the subpath of Q_{26} between u_2 and p_1, and Q' is the subpath of P_3 between a_3 and q_2. This completes the proof. ∎

(9.13) *Assuming* (9.1)–(9.3), *let H be a minimal frame on $(a_1, \ldots, a_5)$, and let S be a side of H. There do not exist disjoint H-paths P, Q with ends p_1p_2 and q_1q_2 respectively, such that*

(i) *p_1, q_1, p_2 lie in $V(S)$ in order on S, and $q_2 \in V(R_k) - V(S)$ for some k with $S \subseteq R_k$ $(1 \leq k \leq 7)$, and*

(ii) *there is a path R in G from $V(P) - \{p_1, p_2\}$ to $V(R_k \cup Q) - V(S)$ with no internal vertex in $V(H \cup P \cup Q)$.*

Proof. First we prove that

(1) *There do not exist such P, Q, R with $Q \cap R$ null.*

For suppose such P, Q, R exist with $Q \cap R$ null. Let R have ends $a \in V(P) - \{p_1, p_2\}$ and $b \in V(R_k) - V(S)$. Let us examine the order of occurrence of p_1, p_2, q_1, q_2, b in R_k. We may assume that p_1, q_1, p_2, q_2 occur in R_k in order, and since $b \notin V(S)$ and $p_1, p_2 \in V(S)$, it follows that the order of the five vertices is one of

$$\begin{gathered}
p_1, q_1, p_2, q_2, b\\
p_1, q_1, p_2, b, q_2\\
b, p_1, q_1, p_2, q_2.
\end{gathered}$$

In the first and third cases let $j = 2$, and in the second case let $j = 1$. Then the vertices q_1, q_2, p_j, b occur in the orders

$$\begin{gathered}
q_1, p_j, q_2, b\\
p_j, q_1, b, q_2\\
b, q_1, p_j, q_2
\end{gathered}$$

in the three cases. But there are disjoint H-paths with ends q_1q_2 and p_jb respectively, and so from (9.12), there is a side S' containing ≥ 3 of q_1, q_2, p_j, b. Now $S' \neq S$ since $b, q_2 \notin V(S)$; and so $q_1 \notin V(S')$ since S is the only side containing q_1. Consequently, $\{q_2, p_j, b\} \subseteq V(S')$, and so $V(S \cap S') = \{p_j\}$. Thus $j = 2$ since p_1, q_1, p_2, q_2 are in order in R_k, and so p_1, q_1, p_2, q_2, b are in order in R_k. Let H' be the minimal frame obtained from H by deleting the edges and internal vertices of the subpath of S between p_1 and p_2, and adding P; then R is an H'-path and so is the union of Q and the subpath of S between p_1 and q_1, and they are disjoint, contrary to (9.12) applied to H'. This proves (1).

From (1) it follows that if P,Q,R,S exist then $P \cup Q \cup R \cup S$ is a tripod with feet s_1, s_2, q_2 and with no other vertex in $Z = V(H) - (V(S) - \{s_1, s_2\})$, where S has ends s_1, s_2. Consequently, by (3.3) we may choose P,Q,R,S and H so that there is a path R' from $V(P) - V(S)$ to $V(H) - V(S)$ disjoint from $V(Q \cup S)$. But by (9.9), R' has both ends in $V(R_k)$, contrary to (1). ∎

A frame H on $(a_1,\dots,a_5)$ in G is *secure* if, with the usual notation,

(a) it is minimal

(b) each side is induced (that is, every edge of G with both ends in the side is an edge of the side), and

(c) for each H-flap C, there is no side S of H with $N(C) \subseteq V(S)$.

(9.14) *Assuming* (9.1)–(9.3), *if there is a frame on* $(a_1,\dots,a_5)$ *then there is a secure frame on* $(a_1,\dots,a_5)$.

Proof. Let H be a minimal frame on $(a_1,\dots,a_5)$. An H-flap C is *good* (with respect to H) if $N(C) \not\subseteq V(S)$ for each side S of H; and *bad* otherwise. Let $C_1,\dots,C_r$ be the good H-flaps, ordered with

$$|V(C_1)| \geq |V(C_2)| \geq \dots \geq |V(C_r)|,$$

and let $D_1,\dots,D_s$ be the bad H-flaps, ordered with

$$|V(D_1)| \geq |V(D_2)| \geq \dots \geq |V(D_s)|.$$

The sequence

$$(|V(C_1)|,\dots,|V(C_r)|,|V(D_1)|,\dots,|V(D_s)|$$

is called the *signature* of H. Among all minimal frames, choose H so that its signature is maximum, using the lexicographic order of signatures; that is, if H has signature $(\alpha_1,\dots,\alpha_n)$, then no frame has signature $(\beta_1,\dots,\beta_m)$ where for some $j \leq \min(n, m-1)$,

(i) $\alpha_i = \beta_i$ for $1 \leq i \leq j$, and

(ii) either $n = j$ or $\alpha_{j+1} < \beta_{j+1}$.

We shall show that H is secure.

Let S be a side of H, and suppose that $a, b \in V(S)$ are adjacent in G but not in S. Let H' be obtained from H by replacing the subpath of S between a and b by the edge ab; then H' is a minimal frame. Each H-flap is a subset of an H'-flap, and each good H-flap is a subset of a good H'-flap. Consequently, the signature of H' is greater than that of H, a contradiction. This proves that each side is induced.

Suppose that there is a bad H-flap, and choose a bad H-flap C with $|V(C)|$ minimum. Thus $|V(C)|$ is the last term of the signature of H. Choose a side S with $N(C) \subseteq V(S)$. Let $a, b \in V(S) \cap N(C)$ be the first and last members of $N(C)$ in S. From (9.3), there is an H-flap $C' \neq C$ such that some member of $N(C')$ lies in S strictly between a and b. Let H' be obtained from H by replacing the subpath of S between a and b by a path between a and b with all its internal vertices in C. Then H' is minimal. Moreover, every H-flap except C is a subset of an H'-flap, and every good H-flap is a subset of a good H'-flap. Since C' is a proper subset of an H'-flap, it follows that the signature of H' is greater than that of H, a contradiction. Thus there is no bad H-flap, and so H is secure. ∎

Assuming (9.1)–(9.3), let H be a secure frame on $(a_1,\dots,a_5)$. It follows from (9.9) and the definition of "secure" that for every H-flap C, either $N(C) \subseteq V(R_{17} \cup R_{56})$, or there is a unique $i\,(1 \le i \le 7)$ such that $N(C) \subseteq V(R_i)$. We define G_0 to be the subgraph of G induced on

$$V(R_{17} \cup R_{56}) \cup \bigcup (C : C \text{ is an } H\text{-flap with } N(C) \subseteq V(R_{17} \cup R_{56}))$$

and for $1 \le i \le 7$ we define G_i to be the subgraph of G induced on

$$V(R_i) \cup \bigcup (C : C \text{ is an } H\text{-flap with } N(C) \subseteq V(R_i) \text{ and } N(C) \not\subseteq V(R_{17} \cup R_{56})).$$

Then every edge of G not in H belongs to exactly one of $G_0, G_1, \dots, G_7$, and

$$G = G_0 \cup G_1 \cup \dots \cup G_7.$$

(9.15) *Assuming* (9.1)–(9.3), *let* H *be a secure frame on* $(a_1,\dots,a_5)$, *and let* $1 \le i \le 7$. *Then* G_i *(defined as above) can be drawn in a disc with* R_i *drawn on the boundary.*

Proof. There is no (≤ 3)-separation (X,Y) of G_i with $V(R_i) \subseteq X$ and $|Y - X| \ge 2$, for otherwise (X',Y) would violate (9.3), where

$$X' = X \cup \bigcup (V(G_j) : 0 \le j \le 7, j \ne i).$$

Consequently, by (2.4) it suffices to show that there do not exist disjoint R_i-paths P,Q in G_i with ends p_1p_2 and q_1q_2 respectively, so that p_1,q_1,p_2,q_2 occur in R_i in order. Suppose then that there exist such P,Q. By (9.12) there is a side S containing ≥ 3 of p_1,q_1,p_2,q_2, and S is a subpath of R_i. We may assume that p_1,q_1,p_2 lie in S in order.

(1) $q_2 \in V(S)$.

For suppose not; then $q_2 \in V(R_i) - V(S)$. Since $P \subseteq G_i$ and H is secure and $V(P) \ne \{p_1,p_2\}$ (because S is induced) there is a path R of G from $V(P) - \{p_1,p_2\}$ to $V(R_i \cup Q) - V(S)$, with no internal vertex in $V(H \cup P \cup Q)$, contrary to (9.13). This proves (1).

By (1) we may assume that p_1,q_1,p_2,q_2 are in $V(S)$ in order. Since P,Q belong to G_i and H is secure there is a minimal path R from $V(P \cup Q) - \{p_1,p_2,q_1,q_2\}$ to $V(R_i) - V(S)$ with no internal vertex in $V(H)$. We may assume that R has one end $a \in V(Q) - \{q_1,q_2\}$ and the other $b \in V(R_i) - V(S)$. Let Q' be the path in $Q \cup R$ from q_1 to b; then P,Q' violate (1), a contradiction. The result follows. ∎

Let $z_1,\dots,z_9$ be distinct vertices of a graph J which can be drawn in a disc with $z_1,\dots,z_9$ on the boundary in order. Let J_1 be obtained from J by identifying z_6 with z_8; let J_2 be obtained from J by identifying z_7 with z_9; and let J_3 be obtained from J by identifying z_6 with z_8 and z_7 with z_9. If $K \in \{J,J_1,J_2,J_3\}$, we call $(K,z_1,\dots,z_5)$ a *twisted* graph with *twist* the set of at most four vertices corresponding to z_6,z_7,z_8,z_9. Finally, we deduce the main result of this section, the following.

(9.16) *Assuming (9.1)–(9.3), if there is a frame on $(a_1,\dots,a_5)$ then there is a (≤ 4)-separation (X,Y) of G with $a_1,\dots,a_5 \in X$, such that $((G\,|\,X) \setminus E(G\,|\,X \cap Y), a_1,\dots,a_5)$ is a twisted graph with twist $X \cap Y$.*

Proof. Let H be a secure frame; this exists by (9.14). Let $X = \bigcup(V(G_i) : 1 \leq i \leq 7)$ and $Y = V(G_0)$. By (9.4), (X,Y) is a (≤ 4)-separation of G and the result follows from (9.15). ∎

10. 7-connectivity of Hadwiger graphs

In this section we combine the results of sections 8 and 9 to close the gap left by (7.16). We need the following lemma. ($\mathcal{P}_1,\dots,\mathcal{P}_{12}$ were defined in section 8.)

(10.1) *Let G be a graph, let $v_1,\dots,v_6 \in V(G)$ be distinct, so that $\mathcal{P}_1,\dots,\mathcal{P}_{12}$ are infeasible in G, and for $1 \leq k \leq 6$ there is no turkey in G on $(v_1,\dots,\hat{v}_k,\dots,v_6)$. For $1 \leq i \leq 5$ let v_iv_{i+1} be adjacent. Let H be obtained from G by adding five new vertices $a_1,\dots,a_5$, where a_i has neighbours v_i and v_{i+1} $(1 \leq i \leq 5)$. Then $\{a_1,a_3,a_5\}$, $\{a_2,a_4\}$ is infeasible in H and there is no turkey on $(a_1,\dots,a_5)$ in H.*

Proof. We denote the partition $\{a_1,a_3,a_5\}$, $\{a_2,a_4\}$ by $\mathcal{P}$. Suppose first that $\mathcal{P}$ is strongly feasible in H. Thus, there is a triad T on H with feet a_1,a_3,a_5 and a path S with ends a_2,a_4 such that $S \cap T$ is null. Choose S,T with $V(S \cup T)$ minimal. For $1 \leq i \leq 5, a_i$ has valency 1 in $S \cup T$; let its neighbour in $S \cup T$ be b_i. Then $b_i \in \{v_i,v_{i+1}\}$, and $b_1,\dots,b_5$ are all distinct. Either $b_3 = v_3$ or $b_3 = v_4$, so from the symmetry we may assume that $b_3 = v_3$, and hence $b_1 = v_1$ and $b_2 = v_2$.

Suppose first that $b_4 = v_4$. Then $v_5 \notin V(S)$ by the minimality of $V(S \cup T)$, and $v_6 \notin V(S)$ since $\mathcal{P}_{12}$ is not feasible in G. Hence $v_5 \in V(T)$ since $\mathcal{P}_5$ is not feasible, and $v_6 \in V(T)$ since $\mathcal{P}_6$ is not feasible. By the minimality of $V(S \cup T)$, v_5 and v_6 both belong to the path of T between v_1 and v_3, and hence one of $\mathcal{P}_{10},\mathcal{P}_{11}$ is feasible, a contradiction. Thus $b_4 \neq v_4$, and so $b_4 = v_5$ and $b_5 = v_6$. Then $v_4 \notin V(S)$ by the minimality of $V(S \cup T)$, and $v_4 \in V(T)$ since $\mathcal{P}_4$ is not feasible. By the minimality of $V(S \cup T)$, v_3 and v_4 both belong to the path of T between v_1 and v_6, and hence one of $\mathcal{P}_7,\mathcal{P}_8$ is feasible, a contradiction. This proves that $\mathcal{P}$ is not strongly feasible in H.

Now suppose that $\mathcal{P}$ is feasible in H. Since it is not strongly feasible, we may write $\{a_1,a_3,a_5\} = \{c_1,c_2,d\}$ in such a way that there are three paths P_1,P_2,Q of H, disjoint except for their ends, where Q has ends a_2a_4, and P_i has ends c_id $(i = 1,2)$. Suppose first that $d = a_1$; then we may assume that $c_1 = a_3$ and $c_2 = a_5$. Consequently, $v_1,v_2 \in V(P_1 \cup P_2)$, and so $v_3 \in V(Q)$, $v_4 \in V(P_1)$, $v_5 \in V(Q)$ and $v_6 \in V(P_2)$. But then $\mathcal{P}_9$ is feasible in G if $v_1 \in V(P_1)$ and $\mathcal{P}_{11}$ is feasible in G if $v_2 \in V(P_1)$, in either case a contradiction. Thus $d \neq a_1$, and by symmetry $d \neq a_5$; hence $d = a_3$, and we may assume that $c_1 = a_1$ and $c_2 = a_5$. Hence $v_3,v_4 \in V(P_1 \cup P_2)$, and so $v_2,v_5 \in V(Q)$, $v_1 \in V(P_1)$ and $v_6 \in V(P_2)$. Then $\mathcal{P}_7$ is feasible in G if $v_3 \in V(P_1)$ and $\mathcal{P}_8$ is feasible in G if $v_4 \in V(P_1)$, in either case a contradiction. This proves that $\mathcal{P}$ is not feasible in H.

Now suppose that $\{X_1,X_2,X_3,X_5,X_0\}$ is a turkey in H on $(a_1,\dots,a_5)$, with $a_1,a_4 \in X_1$, $a_2 \in X_2, a_3 \in X_3, a_5 \in X_5$.

(1) $|X_i| \geq 2$ *for* $i=1,2,3,5$.

This is trivial for $i=1$ since $a_1, a_4 \in X_1$. Suppose first that $|X_2|=1$, and hence $X_2=\{a_2\}$. Since X_2X_0 and X_2X_5 are adjacent, it follows that one of v_2, v_3 is in X_0 and the other is in X_5. Consequently $v_1 \in X_1$. Since $v_3 \notin X_3$ and X_0X_3, X_3X_5 are both adjacent, it follows that $v_4 \in X_0 \cup X_3 \cup X_5$, and hence $v_5 \in X_1$ and $v_6 \in X_5$. If $v_2 \in X_0$, then $v_3 \in X_5$ and $v_4 \in X_0 \cup X_3$, and so $\mathcal{P}_{10}$ is feasible, via $X_1, X_0 \cup X_3, X_5$. On the other hand, if $v_2 \notin X_0$, then $v_2 \in X_5, v_3 \in X_0, v_4 \in X_3 \cup X_5$ and $\mathcal{P}_{12}$ is feasible, via $X_0 \cup X_1, X_3 \cup X_5$. This shows that $|X_2| \geq 2$. Now suppose that $|X_3|=1$; then $v_1, v_5 \in X_1, v_2 \in X_2$, $v_6 \in X_5$, and one of v_3, v_4 is in X_0, and the other is in X_5. If $v_3 \in X_0$ then $\mathcal{P}_{12}$ is feasible, via $X_1 \cup X_0, X_2 \cup X_5$; and if $v_3 \in X_5$ then $\mathcal{P}_{10}$ is feasible, via $X_1, X_2 \cup X_0, X_5$. Thus $|X_3| \geq 2$. Finally, suppose that $|X_5|=1$. Then $v_1, v_4 \in X_1, v_2 \in X_2$, $v_3 \in X_3$, and one of v_5, v_6 is in X_2 and the other is in X_3. If $v_5 \in X_2$ then $\mathcal{P}_8$ is feasible, via X_1, X_2, X_3; and if $v_5 \in X_3$ then $\mathcal{P}_9$ is feasible via X_1, X_2, X_3. This proves (1).

From (1), it follows that there exist $1 \leq i_1 < i_2 < i_3 < i_4 < i_5 \leq 6$ such that $v_{i_j} \in X_j$ for $j=1,2,3,5$ and $v_{i_4} \in X_1$. Let $Y_j = X_j - \{a_1, \ldots, a_5\}$ $(j=1,2,3,5,0)$. We claim that $\{Y_1, Y_2, Y_3, Y_5, Y_0\}$ is a turkey in G on $(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5})$.

Let $\{i,j\}$ be one of $\{0,1\},\{0,2\},\{0,3\},\{2,5\},\{3,5\}$; we claim that Y_iY_j are adjacent in G. For X_iX_j are adjacent in H, and so we may assume that there exist $u \in X_i, v \in X_j$ which are adjacent in H with $\{u,v\} \not\subseteq V(G)$. By exchanging i and j if necessary we may assume that $u \notin V(G)$, and so $u=a_h$ for some h with $1 \leq h \leq 5$. But $u \in X_i$, and so $h=i$ if $i \neq 1$, and $h \in \{1,4\}$ if $i=1$. Now a_h has only two neighbours in H, namely v_h and v_{h+1}, and so one of these is v, and the other, w say, is in X_i by (1). Hence, v, w are adjacent, since v_hv_{h+1} are adjacent, and so Y_iY_j are adjacent as required.

To complete the proof that $\{Y_1, Y_2, Y_3, Y_5, Y_0\}$ is a turkey, we must show that each Y_i is a fragment of G. Let $i \in \{1,2,3,5,0\}$ and let C be a component of $G \mid Y_i$; and suppose that $C \neq G \mid Y_i$. Let $G \mid (Y_i - V(C)) = D$. Since X_i is a fragment of H, there exists h with $1 \leq h \leq 5$ such that $a_h \in X_i$ and a_h has neighbours in both $V(C)$ and $V(D)$. But the two neighbours of a_h are adjacent, contradicting that C is a component of $G \mid Y_i$. Thus each Y_i is a fragment of G, and so $\{Y_1, Y_2, Y_3, Y_5, Y_0\}$ is a turkey in G on $(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5})$, contrary to the hypothesis. ∎

Now let us apply (10.1) and the results of sections 8 and 9 to our problem.

(10.2) *Let G be a non-apex Hadwiger graph. Then there is no (≤ 6)-separation (A,B) of G with $|A-B|, |B-A| \geq 2$.*

Proof. Suppose that there is a (≤ 6)-separation (A,B) with $|A-B|, |B-A| \geq 2$. Choose it with $|A|$ minimum. By (7.16), $G \mid A \cap B$ is a 5-edge path, with vertices $v_1, \ldots, v_6$, say, in order.

Let G^* be obtained from $G \mid A$ by adding five new vertices $a_1, \ldots, a_5$, where a_i is adjacent to v_i and to v_{i+1} $(1 \leq i \leq 5)$. By (10.1), (8.6) and (8.7), we deduce

(1) *$\{a_1, a_3, a_5\}, \{a_2, a_4\}$ is infeasible in G^*, and there is no turkey in G^* on $(a_1, \ldots, a_5)$ or on $(a_5, \ldots, a_1)$.*

Moreover,

(2) *G^* is simple, and there is no (≤ 3)-separation (X,Y) of G^* with $a_1,\ldots,a_5 \in X \neq V(G^*)$.*

For suppose that (X,Y) is such a separation, and choose it with X maximal. Suppose that $1 \leq i \leq 6$ and $v_i \notin X$. From the symmetry, we may assume that $i < 6$ and that $v_{i+1} \in X$, since one of $v_1,\ldots,v_6$ belongs to X. Now $a_i \in X$, and since $v_i \in Y - X$ it follows that $a_i \in X \cap Y$. Let $X' = X \cup \{v_i\}$, $Y' = Y - \{a_i\}$. Then (X',Y') is a separation of G^*, since $v_i, v_{i+1} \in X'$, and (X',Y') has the same order as (X,Y). From the maximality of X, it follows that $X' = V(G^*)$, and so $X = V(G^*) - \{v_i\}$. But v_i has ≥ 2 neighbours in $A - B$ (from (6.3) and the minimality of A) and ≥ 2 neighbours in $(A \cap B) \cup \{a_1,\ldots,a_5\}$ (actually, ≥ 4 neighbours unless $i = 1$ or 6), and hence v_i has valency ≥ 4 in G^*, a contradiction since (X,Y) has order ≤ 3. This proves that $v_i \in X$ for $1 \leq i \leq 6$. But then $(Y \cap A, (X \cap A) \cup B)$ is a (≤ 3)-separation of G, a contradiction. This proves (2).

(3) *There is a frame in G^* on $(a_1,\ldots,a_5)$.*

Define $u_1 = v_1, u_2 = a_2, u_6 = v_3, u_3 = a_3$, $u_7 = v_4, u_4 = a_4, u_5 = v_6$. Let R_{17}, R_{56} be disjoint paths of $G \mid (A - \{v_2,v_5\})$, where R_{17} has ends v_1v_4 and R_{56} has ends v_3v_6; these exist by (7.2). Let P_2,P_3,P_4 be 1-vertex paths, with vertex a_i $(i = 2,3,4)$; and let P_1, P_5 be 1-edge paths, formed by the edges a_1v_1 and a_5v_6 respectively. Let Q_{12} consist of the edges v_1v_2 and v_2a_2; let Q_{45} consist of the edges a_4v_5 and v_5v_6; and let every remaining Q_{ij} needed for the frame be the 1-edge path formed by the edge u_iu_j. This proves (3).

We deduce from (1), (2), (3) and (9.16) that

(4) *There is a (≤ 4)-separation (X,Y) of G^* with $a_1,\ldots,a_5 \in X$, such that*

$$((G^*|X) \setminus E(G^*|X \cap Y), a_1,\ldots,a_5)$$

is a twisted graph with twist $X \cap Y$.

From the definition of a twist, it follows that $a_1,\ldots,a_5 \notin X \cap Y$, and so $v_1,\ldots,v_6 \in X$. Hence $(A \cap X, A \cap Y)$ is a (≤ 4)-separation of $G \mid A$ with $A \cap B \subseteq A \cap X$, and so $A \subseteq X$; and hence $V(G^*) = X$.

Let $|A - B| = n$, and let e be the number of edges of G with both ends in $A - B$, and f the number with one end in $A - B$ and the other in $A \cap B$. By (5.6), $2e + f \geq 7n - 2$. Since $f \geq 12$ by (6.3) and the minimality of A, it follows that $2e + 2f \geq 7n + 10$; and so $|E(G \mid A)| \geq 7n/2 + 10$, since $G \mid (A \cap B)$ has five edges. Thus $|E(G^*)| \geq 7n/2 + 20$. But

$$((G^*|X) \setminus E(G^*|X \cap Y), a_1,\ldots,a_5)$$

is a twisted graph with twist $X \cap Y$; let $J, z_1,\ldots,z_9$ be as in the definition of twisted graph. Let $|Y| = k$; then $|X \cap Y| = k$, and $k = 2,3$ or 4, and $|E(G \mid Y)| \leq \frac{1}{2}k(k-1)$. Now

$$|V(J)| = |X| + 4 - k = n + 15 - k$$

and J can be drawn in a disc with $z_1,\ldots,z_9$ on the boundary in order. Since z_6,z_7,z_8,z_9 are mutually non-adjacent in J and so are z_1,z_2,z_3,z_4,z_5, it follows that

$$|E(J)| \leq 3|V(J)| - 6 - 13 = 3(n + 15 - k) - 19 = 3n - 3k + 26.$$

But $|E(G^*)| \leq |E(J)| + \frac{1}{2}k(k-1)$, since $X = V(G^*)$, and so

$$|E(G^*)| \leq 3n + 26 - 3k + \frac{1}{2}k(k-1).$$

Since $k = 2,3$ or 4, and so $-3k + \frac{1}{2}k(k-1) \leq -5$, it follows that

$$|E(G^*)| \leq 3n + 21.$$

Yet $|E(G^*)| \geq 7n/2 + 20$, and so $7n/2 + 20 \leq 3n + 21$, that is, $n \leq 2$, contrary to (6.3). The result follows. ∎

11. Forbidden subgraphs

With the aid of (10.2) we now prove the absence of several kinds of subgraph in a non-apex Hadwiger graph. We begin with the following.

(11.1) *Let G be a non-apex Hadwiger graph, and let $X \subseteq V(G)$, with $|E(G \mid X)| = g$ and $|V(G) - X| = n$. Then $|E(G \setminus X)| \geq 3n - 4|X| + 8 + g$.*

Proof. Let $|E(G \setminus X)| = e$, and let f be the number of edges with one end in X and the other in $V(G) - X$. Then by (5.6), $2e + f \geq 7n - 2$. But by (6.1), $e + f + g \leq 4(n + |X|) - 10$. Hence, subtracting, $e - g \geq 3n - 4|X| + 8$, as required. ∎

(11.2) *Let G be a non-apex Hadwiger graph, and let $X \subseteq V(G)$ with $|X| = 3$. Let $v_1, v_2, v_3, v_4 \in V(G) - X$ be distinct. Then there is a 4-cluster $\{X_1, \ldots, X_4\}$ of $G \setminus X$ with $v_i \in X_i$ $(i = 1, \ldots, 4)$.*

Proof. Let $|V(G) - X| = n$. By (11.1), $|E(G \setminus X)| \geq 3n - 4 > 3n - 6$ and hence $G \setminus X$ is non-planar. But $G \setminus X$ is 3-connected and has no 3-separation (A, B) with $|A - B|, |B - A| \geq 2$, by (10.2), and the result follows from (2.6). ∎

(11.3) *Let G be a non-apex Hadwiger graph, let $v \in V(G)$ have valency 6, and let N be the set of neighbours of v. If $G \mid N$ has two disjoint triangles then it has no more edges.*

Proof. Let $N = \{v_1, \ldots, v_6\}$, where v_1, v_3, v_5 are mutually adjacent, and so are v_2, v_4, v_6; and suppose v_5, v_6 are adjacent. By (11.2) there is a 4-cluster $\{X_1, \ldots, X_4\}$ in $G \setminus \{v, v_5, v_6\}$ with $v_i \in X_i$ $(1 \leq i \leq 4)$; but then $\{X_1, X_2, X_3, X_4, \{v\}, \{v_5, v_6\}\}$ is a 6-cluster in G, a contradiction. ∎

Let $F_1, \ldots, F_{10}$ be the graphs shown in figure 5. By an *F_i-subgraph* of G we mean a subgraph of G isomorphic to F_i.

(11.4) *Let G be a non-apex Hadwiger graph. Then for $1 \leq i \leq 5, G$ has no F_i-subgraph.*

Proof. G has no F_1-subgraph by (2.7). Suppose it has an F_2-, F_3-, F_4- or F_5-subgraph. In each case there are seven distinct vertices $x, y, z, v_1, v_2, v_3, v_4$ of G such that xy, yz are adjacent and for $1 \leq i \leq 4$, xv_i are adjacent and either yv_i

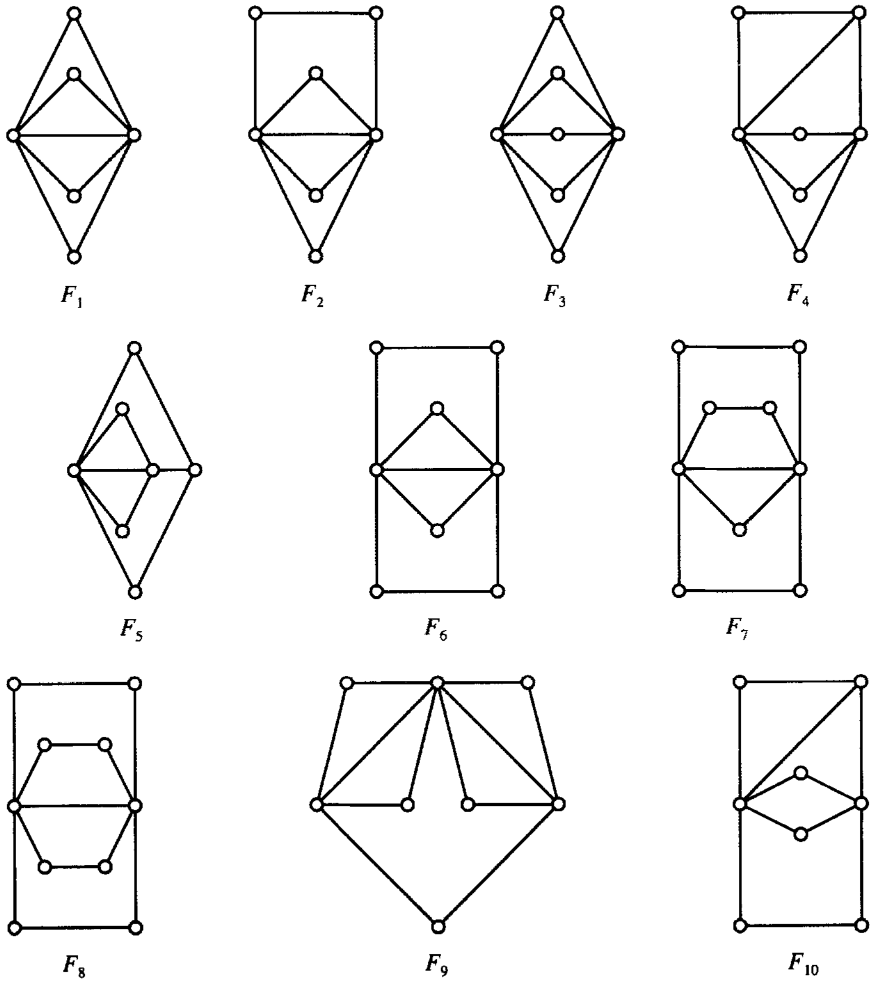

Fig. 5. Forbidden subgraphs

are adjacent or zv_i are adjacent. By (11.2) there is a 4-cluster $\{X_1,\ldots,X_4\}$ in $G\setminus\{x,y,z\}$ with $v_i\in X_i\ (1\leq i\leq 4)$. But then $\{X_1,\ldots,X_4,\{x\},\{y,z\}\}$ is a 6-cluster in G, a contradiction. ∎

If v is a vertex of a graph G, we denote the set of neighbours of v in G by $N(v)$.

(11.5) *Let G be a non-apex Hadwiger graph. Then $|(N(u)\cup N(v))-\{u,v\}|\geq 8$ for any two distinct vertices u, v of G, with equality only if both u and v are 6-valent.*

Proof. Since G has no F_3-subgraph by (11.4), $|N(u) \cap N(v)| \leq 4$. Consequently, if uv are not adjacent,

$$|(N(u) \cup N(v)) - \{u,v\}| = |N(u) \cup N(v)| \geq |N(u)| + |N(v)| - 4 \geq 8,$$

and the result holds. We assume then that uv are adjacent, and so

$$|(N(u) \cup N(v)) - \{u,v\}| = |N(u)| + |N(v)| - |N(u) \cap N(v)| - 2.$$

If $|N(u) \cap N(v)| \leq 2$ the result therefore holds. If $|N(u) \cap N(v)| \geq 3$, then $|N(u) \cap N(v)| = 3$ by (2.7), and $|N(u)| + |N(v)| \geq 14$ by (5.4), and again the result holds. ∎

(11.6) *Let G be a non-apex Hadwiger graph, and let (A,B) be a 7-separation of G with $|A-B| \geq 2$. Then $|A-B| \geq 4$.*

Proof. For all distinct $u,v \in A-B$ we have

$$|(N(u) \cup N(v)) - \{u,v\}| \leq |A| - 2 = |A-B| + 5.$$

Hence by (11.5), $|A-B| \geq 3$. Moreover, if $|A-B| = 3$ then all vertices in $A-B$ are 6-valent by (11.5), contrary to (5.6). ∎

We recall that $\eta(A,B)$ was defined just before (6.4).

(11.7) *Let G be a non-apex Hadwiger graph, and let (A,B) be a 7-separation with $|A-B|, |B-A| \geq 2$. Let $A \cap B = \{v_1,\ldots,v_7\}$. Suppose that either $\eta(A,B) \geq 12$ or every vertex in $A-B$ has valency ≥ 7. Then there is a 4-cluster $\{X_1,\ldots,X_4\}$ in $G \mid (A - \{v_5,v_6,v_7\})$ with $v_i \in X_i$ $(1 \leq i \leq 4)$.*

Proof. We suppose, for a contradiction, that for some (A,B) there is no such 4-cluster, and choose $|A|$ as small as possible.

(1) *There is no 7-separation (A',B') of G with $A' \subseteq A$, $B \subseteq B'$, $|A'-B'| \geq 2$ and $|A'| < |A|$.*

For suppose that (A',B') is such a separation. Let $P_1,\ldots,P_7$ be disjoint paths of $G \mid (A \cap B')$, where P_i has ends v_i and $v_i' \in A' \cap B'$ $(1 \leq i \leq 7)$. (These exist by (10.2).) If $\eta(A,B) \geq 12$ then $\eta(A',B') \geq 12$, while if every vertex in $A-B$ has valency ≥ 7 then every vertex in $A'-B'$ has valency ≥ 7. Consequently, from the minimality of $|A|$, there is a 4-cluster $\{X_1',\ldots,X_4'\}$ in $G \mid (A' - \{v_5',v_6',v_7'\})$ with $v_i' \in X_i'$ $(1 \leq i \leq 4)$. Let $X_i = X_i' \cup V(P_i)$ $(1 \leq i \leq 4)$; then $\{X_1,\ldots,X_4\}$ satisfies the theorem, a contradiction. This proves (1).

We deduce from (1) and (11.6) that

(2) *Every vertex in $A \cap B$ has ≥ 2 neighbours in $A-B$.*

Let $H = G|(A - \{v_5,v_6,v_7\})$.

(3) *There is no trisection (C_1,C_2,D) of H of order 2 with*

$$|(C_i - D) \cap \{v_1,v_2,v_3,v_4\}| = 1 \quad \textit{for} \quad i = 1,2.$$

For suppose that (C_1, C_2, D) is such a trisection, with $v_i \in C_i - D$ $(i=1,2)$ say. Let $C_1 \cap C_2 \cap D = \{a,b\}$, and let $C = C_1 \cup C_2$. Since $(D \cup \{v_5, v_6, v_7\}, C \cup B)$ is a (≤ 7)-separation of G, and $v_1, v_2 \notin D$, it follows from (1) that $|V(G) - (B \cup C)| \leq 1$, that is,

$$|D - \{a, b, v_3, v_4\}| \leq 1.$$

Also, $(C \cup \{v_5, v_6, v_7\}, D \cup B)$ is a (≤ 7)-separation of G, and so either $C \cup \{v_5, v_6, v_7\} = A$ or $|V(G) - (D \cup B)| \leq 1$.

Suppose that $C \cup \{v_5, v_6, v_7\} = A$. Then $D = \{v_3, v_4\}$, since $|C \cap D| = 2$; and hence $C_1 \cap C_2 = D$. Since $|A - B| \geq 4$ by (11.6), we may assume that $|C_1 - B| \geq 2$, and so $(C_1 \cup \{v_5, v_6, v_7\}, D \cup B \cup C_2)$ is a 6-separation of G violating (10.2).

Hence $C \cup \{v_5, v_6, v_7\} \neq A$, and so $|V(G) - (D \cup B)| \leq 1$, that is, $|C - \{a, b, v_1, v_2\}| \leq 1$. But $|D - \{a, b, v_3, v_4\}| \leq 1$, and by (11.6), $|A - B| \geq 4$, and so $|A - B| = 4$, and $a, b \in A - B$, and $C = \{a, b, v_1, v_2, c\}$ and $D = \{a, b, v_3, v_4, d\}$, where $A - B = \{a, b, c, d\}$; and we may assume that $C_1 = \{v_1, a, b, c\}$, $C_2 = \{v_2, a, b\}$. It follows that c is not adjacent to v_2, and hence c is 6-valent, with neighbours v_5, v_6, v_7, a, b, and v_1. If d is adjacent to all of v_5, v_6, v_7, a, b then G has an F_3-subgraph, contrary to (11.4). Thus d is also 6-valent, adjacent to v_3, v_4 and to four of v_5, v_6, v_7, a, b.

Suppose that ab are not adjacent. Since c, d are 6-valent, it follows that a, b have valency ≥ 7, and hence have ≥ 5 common neighbours in $\{v_1, \ldots, v_7, c, d\}$ contrary to (11.4). Thus ab are adjacent. Since the edge ab is in ≤ 3 triangles, and a, b have valency ≥ 7, it follows that a, b have valency 7, that there are exactly three vertices adjacent to both a and b, and that each of $v_1, \ldots, v_7, c, d$ is adjacent to at least one of a and b. In particular we may assume that ≥ 2 of v_5, v_6, v_7 are adjacent to a. But then the edge ac is in ≥ 3 triangles, contrary to (5.4). This proves (3).

(4) *There is no (≤ 3)-separation (C, D) of H with $v_1, \ldots, v_4 \in C$ and $|D - C| \geq 2$.*

For if (C, D) is such a separation then $(B \cup C, D \cup \{v_5, v_6, v_7\})$ is a (≤ 6)-separation of G, and

$$|D \cup \{v_5, v_6, v_7\} - (B \cup C)| = |D - C| \geq 2,$$

contrary to (10.2).

From (2), (3), (4) and (2.6), we deduce

(5) *H can be drawn in a plane with v_1, v_2, v_3, v_4 all incident with the infinite region.*

Moreover, we have

(6) $\eta(A, B) \geq 11$.

For suppose not. Choose $v \in B - A$ with valency ≥ 7; this is possible by (5.6) and (11.6). Then v is joined to $A \cap B$ by seven paths, disjoint except for v, by (10.2); and so by (6.4) there is a separation (C, D) of $G \mid B$ with $C \cap D = \{v\}$ and $|C \cap A|, |D \cap A| \geq 2$. From the symmetry, we may assume that $|D \cap A| \geq 4$ and hence $|C \cap A| \leq 3$. Thus $(C, D \cup A)$ is a (≤ 4)-separation of G, and so $D \cup A = V(G)$. But $(D, C \cup A)$ is a (≤ 6)-separation of G, since $A \cap C \cap D = \emptyset$; and so $|D - (C \cup A)| \leq 1$. Hence $|D - A| \leq 2$, and so $|B - A| \leq 2$ contrary to (11.6). This proves (6).

(7) *There are ≤ 4 vertices in $A-B$ with a neighbour in $\{v_1,\ldots,v_4\}$, and $v_1,\ldots,v_4$ each have exactly two neighbours in $A-B$.*

For suppose not; let us apply (6.5), with $k=7$ and $Z=\{v_1,v_2,v_3,v_4\}$. With δ,ε as in (6.5), $\varepsilon=1$ by (2); and either $\delta=1$ or $\eta(A,B)\geq 12$, and so by (6), $\delta+\eta(A,B)\geq 12$. Then (6.5)(i) is false, by (2); (6.5)(ii) is false, since $\delta+\eta(A,B)\geq 12$; (6.5)(iii) is false since G has no F_3-subgraph, by (11.4); and (6.5)(iv) is false, by (5). This is a contradiction, and so (7) holds.

Let J be the subgraph of G with $V(J)=(A-B)\cup\{v_1,v_2,v_3,v_4\}$ and edges the edges of G with at least one end in $A-B$ and with both ends in $V(J)$.

(8) *J is 2-connected.*

For suppose that (C,D) is a (≤ 1)-separation of J with $C,D\neq V(J)$. Then

$$(D\cup B, C\cup\{v_5,v_6,v_7\})$$

is a separation of G of order

$$|C\cap D|+7-|B\cap D|.$$

If $D\cup B=V(G)$, choose $v\in V(H)-D$; then $v\in\{v_1,v_2,v_3,v_4\}$. By (2), v has ≥ 2 neighbours in $A-B$, and both are in C since $v\in V(H)-D$; and so $|C\cap D|\geq 2$, a contradiction. Hence $D\cup B\neq V(G)$. Consequently, $(D\cup B, C\cup\{v_5,v_6,v_7\})$ has order ≥ 6, and so $|B\cap D|\leq 1+|C\cap D|\leq 2$. Similarly $|B\cap C|\leq 2$, and so $|B\cap D|=|B\cap C|=2$ and $|C\cap D|=1$. Consequently, $(D\cup B, C\cup\{v_5,v_6,v_7\})$ has order 6, and so $|V(G)-(D\cup B)|=1$, that is, $|C-(D\cup B)|=1$. Similarly $|D-(C\cup B)|=1$, and so $|A-B|\leq 3$, contrary to (11.6). This proves (8).

Let N be the set of vertices in $A-B$ with a neighbour in $\{v_1,\ldots,v_4\}$. Take a drawing of H as in (5); since J is a subgraph of H, this yields a drawing of J. By (8) there is a circuit C bounding the infinite region of the latter. By (5), $\{v_1,\ldots,v_4\}\in V(C)$, and by (7), $V(C)=N\cup\{v_1,\ldots,v_4\}$ and $|N|=4$, since $\{v_1,\ldots,v_4\}$ is stable in J. Let the vertices of C be $v_1,a_1,v_2,a_2,v_3,a_3,v_4,a_4$ in order.

(9) $|A-B|\geq 6$.

By (5.6), we may assume without loss of generality that a_1 is not 6-valent. If $A-B=\{a_1,a_2,a_3,a_4\}$ then by (7),

$$(N(a_1)\cup N(a_2))-\{a_1,a_2\}\subseteq\{a_3,a_3,v_1,v_2,v_3,v_5,v_6,v_7\},$$

contrary to (11.5). Thus $|A-B|\geq 5$. If $A-B=\{a_1,a_2,a_3,a_4,a\}$ then by (7)

$$(N(a_1)\cup N(a))-\{a_1,a\}\subseteq\{a_2,a_3,a_4,v_1,v_2,v_5,v_6,v_7\},$$

contrary to (11.5). This proves (9).

Now $(A-\{v_1,v_2,v_3,v_4\}, B\cup N)$ is a 7-separation of G, and $|(A-\{v_1,v_2,v_3,v_4\})-(B\cup N)|\geq 2$ by (9). This contradicts (1), and the result follows. ∎

(11.8) *Let G be a non-apex Hadwiger graph, and let (A,B) be a 7-separation with $|A-B|,|B-A|\geq 2$. Then $G\,|\,A\cap B$ has no circuit of length 4 or 5.*

Proof. Suppose that $G\,|\,A\cap B$ has a circuit of length 4 or 5.

(1) *$\eta(A,B) \geq 12$ and $\eta(B,A) \geq 12$.*

For by (11.6), there exists $v \in A - B$ with valency 7, and hence there exist seven paths $P_1,\ldots,P_7$ of $G \mid A$ from v to $A \cap B$, disjoint except for v. Suppose that (C,D) is a separation of $G \mid A$ with $C \cap D = \{v\}$ and $|C \cap B|, |D \cap B| \geq 2$. Since $(C, B \cup D)$ is a separation of G of order

$$|C \cap D| + 7 - |D \cap B| \leq 6$$

it follows that $|C - (B \cup D)| \leq 1$ and similarly $|D - (B \cup C)| \leq 1$. Hence $|A - B| \leq 3$, contrary to (11.6). Thus there is no such (C,D), and so the claim follows from (6.4).

Let $A \cap B = \{v_1,\ldots,v_7\}$. From (11.7) and (1), there is a 4-cluster $\{X_1, X_3, X_6, X_7\}$ in $G \mid (A - \{v_2, v_4, v_5\})$ with $v_i \in X_i$ $(i = 1,3,6,7)$. Similarly there is a 4-cluster $\{Y_2, Y_4, Y_6, Y_7\}$ in $G \mid (B - \{v_1, v_3, v_5\})$ with $v_i \in Y_i$ $(i = 2,4,6,7)$.

It follows that not all of $v_1v_2, v_2v_3, v_3v_4, v_4v_1$ are adjacent; for if they are then

$$\{X_1, Y_2, X_3, Y_4, X_6 \cup Y_6, X_7 \cup Y_7\}$$

is a 6-cluster in G, a contradiction. Similarly not all of $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1$ are adjacent; for if they are then

$$\{X_1, Y_2, X_3, Y_4 \cup \{v_5\}, X_6 \cup Y_6, X_7 \cup Y_7\}$$

is a 6-cluster in G, a contradiction. The result follows. ∎

We need the following lemma.

(11.9) *Let $e_1,\ldots,e_k$ be mutually non-adjacent edges of a simple graph G. Let T be the number of triangles of G containing one of $e_1,\ldots,e_k$, and let S be the number of induced circuits of length 4 containing two of $e_1,\ldots,e_k$. Let H be obtained from G by contracting $e_1,\ldots,e_k$ and deleting any multiple edges. Then*

$$|E(H)| \geq |E(G)| - k - S - T.$$

Proof. Let J be the graph obtained from G by contracting $e_1,\ldots,e_k$; then J is loopless. For each $v \in V(J)$ let Z_v be the set of one or two vertices of G corresponding to v. Let $u, v \in V(J)$ be distinct; we claim

(1) *The number of edges of J with ends u, v is at most one more than the number of induced circuits C of G with $V(C) \subseteq Z_u \cup Z_v$.*

For let the number of edges of J with ends u, v be r, and let there be s induced circuits of G of length 4 and t of length 3 with vertex set in $Z_u \cup Z_v$. If $r \leq 1$ then $s = t = 0$; if $r = 2$ then $s + t = 1$; if $r = 3$ then $s = 0$ and $t = 2$; and if $r = 4$ then $s = 0$ and $t = 4$. In each case $r \leq s + t + 1$, as required.

Now, by summing the inequality of (1) over all adjacent pairs u, v of vertices of J, we deduce that

$$|E(J)| \leq |E(H)| + S + T.$$

But $|E(J)| = |E(G)| - k$, and the result follows. ∎

(11.10) *Let G be a non-apex Hadwiger graph. Then G has no F_6-, F_7- or F_8-subgraph.*

Proof. Suppose that G has such a subgraph K say. We refer to the three cases when K is an F_6-, F_7- or F_8-subgraph, as cases (i), (ii) and (iii) respectively. Let K have vertex set $\{x_1,x_2,v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8\}$, where in case (i) $v_5 = v_6$ and $v_7 = v_8$, in case (ii) $v_7 = v_8$, and otherwise these vertices are all distinct, and x_1 has neighbours x_2,v_1,v_3,v_5,v_7, and x_2 has neighbours x_1,v_2,v_4,v_6,v_8, and v_1v_2 are adjacent, and v_3v_4 are adjacent, and v_5v_6 are adjacent (except in case (i), when $v_5 = v_6$) and v_7v_8 are adjacent (except in cases (i) and (ii), when $v_7 = v_8$).

Let H be obtained from $G \setminus \{x_1,x_2\}$ by contracting the edges v_1v_2, v_3v_4, v_5v_6 (except in case (i)) and v_7v_8 (except in cases (i) and (ii)), forming vertices w_1,w_2,w_3,w_4. (In case (i) we take $w_3 = v_5$ and $w_4 = v_7$, and in case (ii) we take $w_4 = v_7$.)

(1) *There is no 4-cluster $\{X_1,\ldots,X_4\}$ in H with $w_i \in X_i$ $(1 \le i \le 4)$.*

For suppose that $\{X_1,\ldots,X_4\}$ is such a 4-cluster. For $1 \le i \le 4$, let $X_i' = (X_i - \{w_i\}) \cup \{v_{2i-1},v_{2i}\}$; then $\{X_1',X_2',X_3',X_4', \{x_1\},\{x_2\}\}$ is a 6-cluster in G, a contradiction.

(2) *There is no trisection (A_1,A_2,B) of H of order 2 with*

$$|(A_i - B) \cap \{w_1,w_2,w_3,w_4\}| = 1 \qquad (i = 1,2).$$

For suppose that (A_1,A_2,B) is such a trisection. Let $A_1 \cap A_2 \cap B = \{a,b\}$. Let

$$A_1' = (A_1 - \{w_1,w_2,w_3,w_4\}) \cup \bigcup(\{v_{2i-1},v_{2i}\} : 1 \le i \le 4, w_i \in A_1)$$

and define A_2', B' similarly. Then (A_1',A_2',B') is a trisection of $G \setminus \{x_1,x_2\}$ of order ≤ 4. In particular,

$$(A_1' \cup \{x_1,x_2\}, A_2' \cup B' \cup \{x_1,x_2\})$$

is a (≤ 6)-separation of G. Since $A_1 - \{a,b\}$ and $A_2 - \{a,b\}$ both contain members of $\{w_1,\ldots,w_4\}$ and hence are non-empty, it follows that $A_1' \cup \{x_1,x_2\} \ne V(G)$ and $A_2' \cup B' \cup \{x_1,x_2\} \ne V(G)$. Hence this separation has order exactly 6, and so $a,b \in \{w_1,w_2,w_3,w_4\}$, and $a,b \ne w_4$ in cases (i) and (ii) and $a,b \ne w_3$ in case (i). Thus we may assume that $a = w_1, b = w_2$. Let $Z = \{v_1,v_2,v_3,v_4,x_1,x_2\}$. Since $(A_1' \cup \{x_1,x_2\}, A_2' \cup B' \cup \{x_1,x_2\})$ has order 6 it follows that either $|A_1' - Z| \le 1$ or $|(A_2' \cup B') - Z| \le 1$. Similarly either $|A_2' - Z| \le 1$ or $|(A_1' \cup B') - Z| \le 1$. We may therefore assume that $|A_1' - Z| \le 1$. Since $A_1 - \{a,b\}$ contains one of w_3,w_4, it follows that we may assume that $w_4 \in A_1 - \{a,b\}$ and $v_7 = v_8$, and in case (iii), this is a contradiction. It follows therefore that we are in case (i) or (ii), and so $v_7 = v_8 = w_4$. Since $A_1' - Z = \{v_7\}$, every neighbour of v_7 in G is in Z, and so v_7 is 6-valent in G, and v_7 is adjacent to every vertex in Z. But $G \mid Z$ has ≥ 2 circuits of length 4, contrary to (11.3) and (5.3). This proves (2).

(3) *There is no* (≤ 3)*-separation* (A,B) *of* H *with* $w_1,\ldots,w_4 \in A$, $|B-A| \geq 2$, *and* $|\{w_1,\ldots,w_4\} \cap B| \leq 2$.

For suppose that (A,B) is such a separation. Define

$$A' = (A - \{w_1,\ldots,w_4\}) \cup \{v_1,\ldots,v_8\}$$
$$B' = (B - \{w_1,\ldots,w_4\}) \cup \bigcup(\{v_{2i-1},v_{2i}\} : 1 \leq i \leq 4, w_i \in B).$$

Then $|A' \cap B'| \leq 5$, since $|A \cap B| \leq 3$ and $|\{w_1,\ldots,w_4\} \cap B| \leq 2$. Thus $(A' \cup \{x_1,x_2\}, B' \cup \{x_1,x_2\})$ is a (≤ 7)-separation of G. Now $|B'-A'| = |B-A| \geq 2$, and $|A'-B'| \geq |A-B| \geq 2$ since at least two of $w_1,\ldots,w_4$ are in $A-B$. Therefore $(A' \cup \{x_1,x_2\},\ B' \cup \{x_1,x_2\})$ is a 7-separation of G, and $A \cap B$ contains two of $w_1,\ldots,w_4$. This contradicts (11.8). Hence (3) holds.

From (1), (2), (3) and (2.6), we deduce that H can be drawn in a disc with $w_1,\ldots,w_4$ on the boundary in some order. Let H' be obtained from H by deleting all parallel edges; it follows that $|E(H')| \leq 3n-7$ where $n = |V(H)| = |V(H')|$. For $1 \leq i \leq 4$, define $T_i = 0$ if $v_{2i-1} = v_{2i}$, and otherwise let T_i be the number of triangles of $G \setminus \{x_1,x_2\}$ containing the edge $v_{2i-1}v_{2i}$. For $1 \leq i < j \leq 4$, define $S_{ij} = 1$ if $G \mid \{v_{2i-1},v_{2i},v_{2j-1},v_{2j}\}$ is a circuit of length 4, and otherwise let $S_{ij} = 0$. Let k be 2, 3, 4 in cases (i), (ii), (iii), respectively. By (11.9)

$$|E(H')| \geq |E(G \setminus \{x_1,x_2\})| - k - \sum_{1\leq i\leq 4} T_i - \sum_{1\leq i<j\leq 4} S_{ij}.$$

Hence

$$|E(G \setminus \{x_1,x_2\})| \leq 3n - 7 + k + \sum_{1\leq i\leq 4} T_i + \sum_{1\leq i<j\leq 4} S_{ij}.$$

On the other hand, $|V(G \setminus \{x_1,x_2\})| = n+k$, and so by (11.1),

$$|E(G \setminus \{x_1,x_2\})| \geq 3(n+k) - 8 + 8 + 1 = 3n + 3k + 1.$$

Consequently,

$$\sum_{1\leq i\leq 4} T_i + \sum_{1\leq i<j\leq 4} S_{ij} \geq 2k + 8.$$

Now $T_1,T_2,T_3,T_4 \leq 2$, since by (11.4) G has no F_2-subgraph, and so $\sum_{1\leq i<j\leq 4} S_{ij} \geq 2k$. Since $\sum_{1\leq i<j\leq 4} S_{ij} \leq 3$ in cases (i) and (ii), it follows that we are in case (iii), and $k=4$; but then $\sum_{1\leq i<j\leq 4} S_{ij} \leq 6 < 2k$, a contradiction. ∎

(11.11) *Let* G *be a non-apex Hadwiger graph. Then* G *has no* F_9*-subgraph.*

Proof. Suppose that K is such a subgraph, with vertex set x,y,z,w,v_1,v_2,v_3,v_4, where x has neighbours y,z,v_1,v_2,v_3,v_4, and y has neighbours x,v_1,v_2,w, and z has neighbours x,v_3,v_4,w. Let $H = G \setminus \{x,y,z,w\}$.

(1) *There is no 4-cluster* $\{X_1,\ldots,X_4\}$ *in* H *with* $v_i \in X_i$ $(1 \le i \le 4)$.

For if $\{X_1,\ldots,X_4\}$ is such a 4-cluster then

$$\{X_1, X_2, X_3, X_4, \{x\}, \{y,z,w\}\}$$

is a 6-cluster in G, a contradiction.

(2) *There is no trisection* (A_1, A_2, B) *of* H *of order* 2 *with*

$$|(A_i - B) \cap \{v_1, v_2, v_3, v_4\}| = 1 \qquad (i = 1,2).$$

For suppose that (A_1, A_2, B) is such a trisection. Then $(A_1 \cup \{x,y,z,w\}, A_2 \cup B \cup \{x,y,z,w\})$ is a 6-separation of G, and so either $|A_1 - (A_2 \cup B)| \le 1$ or $|(A_2 \cup B) - A_1| \le 1$. Similarly either $|A_2 - (A_1 \cup B)| \le 1$ or $|(A_1 \cup B) - A_2| \le 1$. We may therefore assume that $|A_1 - (A_2 \cup B)| \le 1$. If also $|A_2 - (A_1 \cup B)| \le 1$ then $|(A_1 \cup A_2) - B| = 2$, and then $(A_1 \cup A_2 \cup \{x,y,z,w\}, B \cup \{x,y,z,w\})$ is a 6-separation of G contrary to (10.2), for $|B - (A_1 \cup A_2)| \ge 2$ by (6.2). Thus $|A_2 - (A_1 \cup B)| \ge 2$, and so $|(A_1 \cup B) - A_2| \le 1$. Hence $|B| = 2$ and $A_1 \cup A_2 = V(H)$.

Now $|A_1 - A_2 \cup B| \le 1$, and so we may assume that $A_1 - (A_2 \cup B) = \{v_1\}$. Since v_1 has valency ≥ 6 in G, it follows that $v_1 z$ are adjacent. But then $G \mid \{x,y,z,w,v_1,v_3,v_4\}$ has an F_2-subgraph, a contradiction. This proves (2).

(3) *There is no* (≤ 3)*-separation* (A,B) *of* H *with* $v_1,\ldots,v_4 \in A$ *and* $|B - A| \ge 2$ *and* $|\{v_1,\ldots,v_4\} \cap B| \le 2$.

For if (A,B) is such a separation, then $|A - B| \ge 2$ since $|\{v_1,\ldots,v_4\} \cap B| \le 2$, and $(A \cup \{x,y,z,w\}, B \cup \{x,y,z,w\})$ is a (≤ 7)-separation of G, violating (11.8).

From (1), (2), (3) and (2.6), we deduce that H can be drawn in a disc with $v_1,\ldots,v_4$ on the boundary in some order. Hence $|E(H)| \le 3n - 7$ where $|V(H)| = n$. But by (11.1), $|E(H)| \ge 3n - 16 + 8 + 4$, a contradiction. ∎

(11.12) *Let* G *be a non-apex Hadwiger graph, and suppose that* K *is an* F_{10}*-subgraph of* G. *Then there exists* $v \in V(K)$ *such that* v *has valency* 2 *in* K, *both its neighbours in* K *have valency* ≥ 3 *in* K, *and no vertex in* $V(G) - V(K)$ *is adjacent to* v *in* G.

Proof. Let $V(K) = \{x_1, x_2, x_3, x_4, x_5, v_1, v_2, v_3\}$, where x_1 is 5-valent, x_2 is 4-valent, x_3 is 3-valent, $x_4 x_5$ are adjacent, x_1 has neighbours x_3, v_1, v_2, v_3, x_5, and x_2 has neighbours x_3, v_1, v_2, x_4. Let H be obtained from $G \setminus \{x_1, x_2, x_3\}$ by contracting the edge $x_4 x_5$, forming a vertex w say.

(1) *There is no 4-cluster* $\{X_1,\ldots,X_4\}$ *in* H *with* $v_i \in X_i$ $(1 \le i \le 3)$ *and* $w \in X_4$.

For otherwise

$$\{X_1, X_2, X_3, (X_4 - \{w\}) \cup \{x_4, x_5\}, \{x_1\}, \{x_2, x_3\}\}$$

is a 6-cluster in G, a contradiction.

(2) *H cannot be drawn in the plane with v_1, v_2, v_3, w all incident with the infinite region.*

For suppose it can. Let H' be obtained from H by deleting any parallel edges; then $|E(H')| \leq 3n-7$ where $|V(H)| = n$. Since x_4x_5 is in ≤ 3 triangles by (2.7), it follows that $|E(H)| \leq 3n-4$, and so $|E(G \setminus \{x_1, x_2, x_3\})| \leq 3n-3$. But by (11.1)

$$|E(G \setminus \{x_1, x_2, x_3\})| \geq 3(n+1) - 12 + 8 + 2 = 3n+1$$

since $|V(G) - \{x_1, x_2, x_3\}| = n+1$, a contradiction. This proves (2).

(3) *There is no (≤ 3)-separation (X, Y) of H with $v_1, v_2, v_3, w \in X$ and $|Y - X| \geq 2$ and $|Y \cap \{v_1, v_2, v_3, w\}| \leq 2$.*

For suppose that (X, Y) is such a separation. Let

$$X' = (X - \{w\}) \cup \{x_1, x_2, x_3, x_4, x_5\},$$
$$Y' = \begin{cases} Y \cup \{x_1, x_2, x_3\} & \text{if } w \notin Y \\ (Y - \{w\}) \cup \{x_1, x_2, x_3, x_4, x_5\} & \text{if } w \in Y. \end{cases}$$

Then (X', Y') is a (≤ 7)-separation of G, with $|Y' - X'| \geq 2$ and $|X' - Y'| \geq 2$ since $|Y \cap \{v_1, v_2, v_3, w\}| \leq 2$. By (10.2), (X', Y') has order 7, and so $w \in X \cap Y$; but then $G \,|\, X' \cap Y'$ has a circuit of length 5 (with vertex set $\{x_1, \ldots, x_5\}$) contrary to (11.8). This proves (3).

From (1), (2), (3) and (2.6), there is a trisection (X_1, X_2, Y) of H of order 2 such that $X_1 - Y$ and $X_2 - Y$ both contain exactly one member of $\{v_1, v_2, v_3, w\}$. We may assume that $|X_1| \leq |X_2|$. Define

$$X'_1 = \begin{cases} X_1 \cup \{x_1, x_2, x_3\} & \text{if } w \notin X_1 \\ (X_1 - \{w\}) \cup \{x_1, x_2, x_3, x_4, x_5\} & \text{if } w \in X_1 \end{cases}$$

and define X'_2, Y' similarly. Then $(X'_1 \cup X'_2, Y')$ is a (≤ 6)-separation of G. But $|(X'_1 \cup X'_2) - Y'| \geq 2$ since $X_1 - Y$ and $X_2 - Y$ both contain one of v_1, v_2, v_3, w; and so by (10.2), $|Y' - (X'_1 \cup X'_2)| \leq 1$. Consequently, $|Y| \leq 3$. From (6.2) $|V(H)| \geq 14$, and so $|X_1| + |X_2| + |Y| \geq 18$. Hence $|X_2| \geq 8$, since $|X_1| \leq |X_2|$.

Now $(X'_2, X'_1 \cup Y')$ is a (≤ 6)-separation of G, and it has order 6 only if $w \in X_2 \cap (X_1 \cup Y)$. But $|X'_2 - (X'_1 \cup Y')| \geq 2$ since $|X_2| \geq 8$, and $|(X'_1 \cup Y') - X'_2| \geq 1$ since $X_1 - Y$ contains one of v_1, v_2, v_3, w. By (10.2), $|(X'_1 \cup Y') - X'_2| = 1$, and $w \in X_2 \cap (X_1 \cup Y)$. It follows that $|Y| = 2, |X_1| = 3$, and $X_1 - Y = \{v_i\}$ for some i with $1 \leq i \leq 3$. Since $|(X_2 - Y) \cap \{v_1, v_2, v_3, w\}| = 1$, it follows that $|Y \cap \{v_1, v_2, v_3, w\}| = 2$, and so $Y \subseteq \{v_1, v_2, v_3, w\}$. Since (X'_1, X'_2) is a 6-separation of G (because $Y' \subseteq X'_2$) it follows that every neighbour of v_i in G belongs to

$$X'_1 \cap X'_2 = Y' \cup \{x_1, x_2, x_3\} \subseteq V(K),$$

and the result holds. ∎

12. Finding a perfect matching

In this section we prove that every non-apex Hadwiger graph G has a matching of cardinality $\left\lfloor \frac{1}{2}|V(G)| \right\rfloor$. For that, we need the following.

(12.1) *Let G be a non-apex Hadwiger graph, and let (A,B) be a 7-separation of G with $|A-B|,|B-A| \geq 2$, such that every vertex in $A-B$ has valency ≥ 7. Let $A \cap B = \{v_1,\ldots,v_7\}$, and let $Y_1,\ldots,Y_7 \subseteq B$ be disjoint fragments with $v_i \in Y_i$ $(1 \leq i \leq 7)$. Then there are disjoint fragments $X_1,\ldots,X_7 \subseteq A$ with $v_i \in X_i$ $(1 \leq i \leq 7)$, such that for at least four pairs i, j with $1 \leq i < j \leq 7$, X_iX_j are adjacent and Y_iY_j are not adjacent.*

Proof. Let H be the graph with $V(H) = \{v_1,\ldots,v_7\}$ in which v_iv_j are adjacent if Y_iY_j are adjacent. We may assume that

(1) *H has minimum valency ≥ 3.*

For suppose that v_1 is not adjacent in H to v_4,v_5,v_6,v_7 say. Choose $v \in A-B$; then by hypothesis, v has valency ≥ 7. Take seven paths $P_1,\ldots,P_7$ in $G \mid A$ disjoint except for v, where P_i has ends vv_i. Let $X_1 = V(P_1)$ and $X_i = V(P_i) - \{v\}$ $(2 \leq i \leq 7)$; since X_1X_i are adjacent for i = 4, 5, 6, 7, the result holds.

(2) *For all $Z \subseteq A \cap B$ with $|Z| = 4$ there is a cluster in $G \mid (A-B) \cup Z$ traversing Z.*

This follows from (11.7), since every vertex in $A-B$ has valency ≥ 7.

Let J be the complement of H; that is, $V(J) = A \cap B$, and v_iv_j are adjacent in J if Y_iY_j are not adjacent in G. We may assume that

(3) *If $Z \subseteq A \cap B$ with $|Z| = 4$ then $J|Z$ has ≤ 3 edges.*

Let $Z = \{v_1,\ldots,v_4\}$ say. By (2) there is a cluster $\{X_1,\ldots,X_4\}$ in $G \mid (A - \{v_5,v_6,v_7\})$ with $v_i \in X_i$ $(1 \leq i \leq 4)$. Let $X_i = \{v_i\}$ $(i = 5,6,7)$; then $X_1,\ldots,X_7$ satisfy the theorem, unless $J|Z$ has ≤ 3 edges. This proves (3).

In particular from (3) we deduce

(4) *J has no circuit of length* 4.

Next, we claim

(5) *If (C,D) is a (≤ 3)-separation of H with $C-D, D-C \neq \emptyset$ then (C,D) has order 3 and one of $|C-D|,|D-C| = 1$.*

For suppose that $|C-D|,|D-C| \geq 2$, and choose distinct $a,b \in C-D$ and $c,d \in D-C$. Then a,b are adjacent in J to c,d, contrary to (4). Hence we may assume that $|C-D| = 1, C-D = \{a\}$, say. But by (1), a has valency ≥ 3 in H, and so $|C \cap D| \geq 3$, as required.

(6) *H is planar.*

For if not, then by (5) and [17], H has a 5-cluster $\{Z_1,\ldots,Z_5\}$ say. Let $W_i = \bigcup(Y_j : 1 \leq j \leq 7, v_j \in Z_i)$ for $1 \leq i \leq 5$; then $\{W_1,\ldots,W_5\}$ is a 5-cluster in $G \mid B$, and $W_i \cap A \cap B \neq \emptyset$ for $1 \leq i \leq 5$. Choose $v \in A-B$; then since v has valency ≥ 7, there are by (10.2) seven paths of $G \mid A$ between v and $A \cap B$, disjoint except for v. Hence there is a fragment $W_6 \subseteq A - B$ such that $v_1, \ldots, v_7$ all have neighbours in W_6; but then $\{W_1, \ldots, W_6\}$ is a 6-cluster in G, a contradiction. This proves (6).

(7) *J has no circuit of length* 3.

For suppose that $v_1, v_2, v_3 \in V(J)$ are pairwise adjacent in J. By (6) not all of v_1, v_2, v_3 are adjacent in H to all of v_4, v_5, v_6 and so we may assume that $v_1 v_4$ are adjacent in J. But then $Z = \{v_1, v_2, v_3, v_4\}$ contradicts (3).

(8) *J has no circuit of length* 7.

For if it has such a circuit, then by (3), J is a circuit of length 7; but then its complement H is non-planar contrary to (6).

Our next objective is to show that J has no circuit of length 5. The proof requires two steps. Suppose therefore that $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1$ are non-adjacent in H. Let $K = G \mid (A - \{v_6, v_7\})$. We may assume, by permuting $v_1, \ldots, v_5$, that

(9) *There is a path P of K with ends v_1v_2, and a vertex $v \in V(K) - \{v_1, v_2, v_3, v_5\}$, and three paths P_3, P_4, P_5 of K from v to v_3, v_4, v_5 respectively, such that P_3, P_4, P_5 are mutually disjoint except for v, and each of them is disjoint from P.*

For by (2) there are disjoint paths P, Q of $K \backslash \{v_5\}$ with ends v_1v_2 and v_3v_4, respectively. Suppose that there is a separation (X, Y) of K with $v_5 \in X$, $V(P \cup Q) \subseteq Y$, and $X \cap Y = \{v_1, v_2, v_3, v_4\}$. Then $(X \cup B, Y \cup \{v_6, v_7\})$ is a 6-separation of G, and so $|Y - X| \leq 1$ by (10.2), and hence one of P, Q has no internal vertices, a contradiction since v_1v_2 and v_3v_4 are non-adjacent in H and hence in G. This proves that there is no such (X, Y), and hence there is a path R of K from v_5 to $V(P \cup Q) - \{v_1, v_2, v_3, v_4\}$ with no vertex in $\{v_1, v_2, v_3, v_4\}$. Choose a minimal such path R, with ends v_5, v say. By exchanging v_1, v_2 with v_4, v_3, we may assume that $v \in V(Q) - \{v_3, v_4\}$; but then (9) holds.

Choose v, P, P_3, P_4, P_5 as in (9) with $|E(P_4)|$ minimum. (Note that possibly $v = v_4$ in (9), and so possibly $E(P_4) = \emptyset$.)

(10) *There is a path of K from $V(P_3 \cup P_5)$ to $V(P) - \{v_1, v_2\}$ with no vertex in $\{v, v_1, v_2\}$.*

For if not, there is a separation (C, D) of K with $C \cap D = \{v, v_1, v_2\}$, $V(P) \subseteq C$ and $V(P_3 \cup P_5) \subseteq D$. Then $(C \cup \{v_6, v_7\}, D \cup \{v_6, v_7\})$ is a separation of $G \mid A$, and so

$$(C \cup \{v_6, v_7\}, D \cup \{v_6, v_7\} \cup B)$$

is a separation of G. Its order is

$$2 + |C \cap D| + |(C - D) \cap \{v_4\}| \leq 6;$$

and so $|C - (D \cup B)| \leq 1$. But $C - (D \cup B) \neq \emptyset$ since $V(P) - \{v_1, v_2\} \neq \emptyset$, and so $|C - (D \cup B)| = 1$, and there exists $u \in C - (D \cup B) \subseteq A - B$ with valency 6 in G, contrary to the hypothesis. This proves (10).

Let Q be a minimal path of K from $V(P_3 \cup P_5)$ to $V(P) - \{v_1, v_2\}$ with no vertex in $\{v, v_1, v_2\}$. Let Q have ends $x \in V(P_3 \cup P_5)$ and $y \in V(P) - \{v_1, v_2\}$. From the symmetry, we may assume that $x \in V(P_3)$, and hence $x \in V(P_3)-\{v\}$. Suppose that $Q \cap P_4$ is non-null, and let the minimal subpath of Q from x to $V(P_4)$ be Q'; let Q' have ends x, v'. Let P_3' be the union of Q' and the subpath of P_3 between v_3 and x; let P_4' be the subpath of P_4 between v_4 and v'; and let P_5' be the union of P_5 and the subpath of P_4 between v and v'. Then P, v', P_3', P_4', P_5' satisfy (9), contrary to the minimality of $|E(P_4)|$. This proves that $Q \cap P_4$ is null. Let

$$\begin{aligned} X_1 &= \{v_1\} \\ X_2 &= V(P) - \{v_1\} \\ X_3 &= V(P_3 \cup Q) - \{y, v\} \\ X_4 &= V(P_4) \\ X_5 &= V(P_5) - \{v\} \\ X_6 &= \{v_6\} \\ X_7 &= \{v_7\} \end{aligned}$$

Then $X_1X_2, X_2X_3, X_3X_4, X_4X_5$ are adjacent in G, and the theorem holds. This proves that we may assume (for a contradiction) that

(11) *J has no circuit of length* 5.

It follows that

(12) *J has no circuit of length* 6.

For suppose that $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6$, v_6v_1 are adjacent in J. By (4), (7) and (11), v_7 has valency ≤ 1 in J, and $J|\{v_1,\dots,v_6\}$ is a circuit. But then H is non-planar, contrary to (6). This proves (12).

(13) $|E(J)| = 6$ *and J is a tree.*

For from (4), (7), (8), (11), (12), J has no circuits and hence has ≤ 6 edges. But by (6), $|E(H)| \leq 15$, and yet $|E(J)| + |E(H)| = 21$. Hence $|E(J)| = 6$ and so J is a tree.

Since J is a tree with maximum valency ≤ 3 by (1) and (13), it has a 4-edge path starting from some 1-valent vertex. Thus we may assume that $v_1v_2, v_2v_3, v_3v_4, v_4v_5$ are all adjacent in J, and v_1 is 1-valent in J. Consequently, v_1v_i are adjacent in H for $3 \leq i \leq 7$; and since J has no circuits, v_2v_5 are adjacent in H, and for $i = 3,4,6,7$ v_i is adjacent in H to at least one of v_2, v_5. By (2) there is a 4-cluster $\{X_3, X_4, X_6, X_7\}$ in $G\,|\,(A - \{v_1, v_2, v_5\})$ with $v_i \in X_i$ $(i = 3,4,6,7)$. But then

$$\{Y_1, Y_2 \cup Y_5, X_3 \cup Y_3, X_4 \cup Y_4, X_6 \cup Y_6, X_7 \cup Y_7\}$$

is a 6-cluster in G, a contradiction. ∎

We use (12.1) to prove the following.

(12.2) *Let G be a non-apex Hadwiger graph. Then G has a matching of cardinality $\geq \frac{1}{2}(|V(G)| - 1)$.*

Proof. Suppose that G has no such matching. By Tutte's theorem [16], there exists $Z \subseteq V(G)$ such that $G \backslash Z$ has $\geq n+2$ components (actually, "odd" components, but that will not matter here), where $n = |Z|$. Since $G\backslash Z$ has ≥ 2 components it follows that $n \geq 6$ since G is 6-connected, and so $G\backslash Z$ has $\geq n+2 \geq 8$ components. Choose n distinct components of $G\backslash Z$, with vertex sets $C_1,\ldots,C_n$, such that for $1 \leq i \leq n$ every vertex in C_i has valency ≥ 7 in G. (This is possible by (5.6).)

For $1 \leq i \leq n$ let N_i be the set of vertices in Z with a neighbour in C_i. Let us number $C_1,\ldots,C_n$ so that

$$|N_i| \leq 7 \text{ and } |C_i| = 1 \text{ for } 1 \leq i \leq h$$

$$|N_i| \leq 7 \text{ and } |C_i| > 1 \text{ for } h+1 \leq i \leq m$$

$$|N_i| \geq 8 \text{ for } m+1 \leq i \leq n.$$

Let $Z \cup C_1 \cup \ldots \cup C_h = \{v_1,\ldots,v_{h+n}\}$. We shall prove the following for $h \leq k \leq m$ by induction on k:

(∗) *There exist disjoint fragments $Y_1,\ldots,Y_{h+n} \subseteq Z \cup C_1 \cup \ldots \cup C_k$ with $v_i \in Y_i$ for $1 \leq i \leq h+n$, such that there are at least $4(h+k)$ pairs i, j with $1 \leq i < j \leq h+n$ for which Y_iY_j are adjacent.*

(1) *(∗) is true when $k = h$.*

For each $v \in C_1 \cup \ldots \cup C_h$, let T_v be the number of triangles containing v. Now v is 7-valent, by the choice of $C_1,\ldots,C_h$ and $C_1,\ldots,C_n$; let N be the set of neighbours of v. By (5.2), $G \mid N$ has no stable set of cardinality 4, and so $G \mid N$ has ≥ 3 edges (in fact more). Hence $T_v \geq 3$. By summing over all such v, we deduce that $|T| \geq 3h$, where T is the set of triangles containing a vertex in $C_1 \cup \ldots \cup C_h$. Since each member of T contains an edge of $G \mid Z$, and each such edge is in ≤ 3 triangles by (2.7), it follows that $|E(G \mid Z)| \geq \frac{1}{3}|T| \geq h$. Since each vertex in $C_1 \cup \ldots \cup C_h$ is 7-valent, there are $\geq 8h$ edges with both ends in $Z \cup C_1 \cup \ldots \cup C_h$, and so (∗) holds with $Y_i = \{v_i\}$ $(1 \leq i \leq h+n)$. This proves (1).

Now let us prove (∗) for $h \leq k \leq m$. We assume inductively that $h+1 \leq k \leq m$, and $Y_1,\ldots,Y_{h+n}$ exist as in (∗) with k replaced by $k-1$; and we shall show that they also exist for k. Let $B = V(G) - C_k$, and $A = C_k \cup N_k$. Then (A,B) is a 7-separation of G, and $|A-B| = |C_k| \geq 2$, and $|B-A| \geq 2$ since $n \geq 3$. Since $N_k \subseteq Z$, we may assume that $N_k = \{v_1,\ldots,v_7\}$. By (12.1) there exist disjoint fragments $X_1,\ldots,X_7 \subseteq A$ such that $v_i \in X_i$ for $1 \leq i \leq 7$, and there are ≥ 4 pairs i,j with $1 \leq i < j \leq 7$ for which X_iX_j are adjacent and Y_iY_j are not adjacent. Let $Y'_i = X_i \cup Y_i$ $(1 \leq i \leq 7)$ and $Y'_i = Y_i$ $(8 \leq i \leq h+n)$; then $v_i \in Y'_i$ $(1 \leq i \leq h+n)$, and $Y'_1,\ldots,Y'_{h+n}$ are disjoint fragments in $Z \cup C_1 \cup \ldots \cup C_k$, since $Y_1,\ldots,Y_{h+n}$ are disjoint fragments in $Z \cup C_1 \cup \ldots \cup C_{k-1}$. Since Y_iY_j are adjacent for $\geq 4(h+k-1)$ pairs i,j with $1 \leq i < j \leq h+n$, and since $Y'_iY'_j$ are adjacent for ≥ 4 more pairs, it follows that $Y'_iY'_j$ are adjacent for $\geq 4(h+k)$ pairs i,j and so (∗) holds.

This completes the inductive proof of (∗), and so in particular (∗) holds when $k = m$. For $h+n+1 \leq i \leq h+2n-m$ let $Y_i = C_{i+m-h-n}$. It follows that $Y_1,\ldots,Y_{h+2n-m}$ are disjoint fragments. Since for all j with $m+1 \leq j \leq n$ there are ≥ 8 values of i with $1 \leq i \leq h+n$ such that $v_i \in N_j$, it follows that for all j with $h+n+1 \leq j \leq h+2n-m$ there are ≥ 8 values of i with $1 \leq i \leq h+n$ such that Y_iY_j are adjacent. In total therefore there are at least

$$4(h+m) + 8(n-m) = 4h + 8n - 4m$$

pairs i,j with $1 \le i < j \le h+2n-m$ such that Y_iY_j are adjacent. By (6.1) applied to the graph obtained from G by contracting all edges with both ends in Y_i for some i and deleting parallel edges, since $h+2n-m \ge h+n \ge n \ge 4$, it follows that

$$4h + 8n - 4m \le 4(h + 2n - m) - 10,$$

a contradiction. Thus there is no such Z, as required. ∎

13. Reducible configurations

We use (12.2) for the following.

(13.1) *Let G be a non-apex Hadwiger graph. Then either*

(i) *there are adjacent vertices a, b of valency 7 and 8 respectively, so that the edge ab is in 3 triangles, and neither a nor b is in a 4-clique, or*

(ii) *there are adjacent vertices a, b, both of valency 7, such that the edge ab is in 3 triangles, and at most one of a, b is in a 4-clique, or*

(iii) *there are distinct vertices a, b, c, d of G, such that ab, bd, ac, cd are adjacent and ad, bc are not, the edges ab and ac are both in 2 triangles, and a, b, c all have valency 7 and are in no 4-clique, and either d has valency 7, or d has valency 8 and is in no 4-clique.*

Proof. We denote the valency of a vertex v by $\delta(v)$. Let M be the set of all edges uv of G that are in exactly two triangles and such that $\delta(u) = \delta(v) = 7$ and u,v belong to no 4-clique. Let $t = \lfloor \frac{1}{2}|V(G)| \rfloor$, and let $|V(G)| = 2t + \varepsilon$; thus, $\varepsilon = 0$ or 1. By (12.2) there exist edges $e_1,\dots,e_t$ of G, pairwise with no common end; choose $e_1,\dots,e_t$ with $|\{e_1,\dots,e_t\} \cap M|$ minimum. For $1 \le i \le t$, let T_i be the number of triangles containing e_i. For $1 \le i,j \le t$, let $S_{ij} = 1$ if $i \ne j$ and the subgraph of G induced on the four ends of e_i, e_j is a circuit, and $S_{ij} = 0$ otherwise. Let d_i be the sum of the valencies of the ends of e_i for $1 \le i \le t$, and let d_0 be the number of edges with an end not incident with any of $e_1,\dots,e_t$. (Thus if $\varepsilon = 0$, then $d_0 = 0$.) Now

$$2|E(G)| = d_0 + \sum_{1 \le i \le t} d_i.$$

Let H be obtained from G by contracting $e_1,\dots,e_t$ and deleting any resulting parallel edges. By (11.9)

$$|E(H)| \ge |E(G)| - t - \sum_{1 \le i,j \le t} \frac{1}{2} S_{ij} - \sum_{1 \le i \le t} T_i.$$

Consequently,

$$2|E(H)| \ge d_0 - 2t + \sum_{1 \le i \le t} (d_i - 2T_i) - \sum_{1 \le i,j \le t} S_{ij}.$$

But $t \geq 4$ by (6.2), and $|V(H)| = 2t + \varepsilon - t = t + \varepsilon$, and so from (6.1),

$$|E(H)| \leq 4(t+\varepsilon) - 10.$$

Consequently,

$$8(t+\varepsilon) - 20 \geq d_0 - 2t + \sum_{1\leq i\leq t} (d_i - 2T_i) - \sum_{1\leq i,j\leq t} S_{ij},$$

that is,

$$\sum_{1\leq i\leq t} (d_i - 2T_i - 10) - \sum_{1\leq i,j\leq t} S_{ij} \leq 8\varepsilon - 20 - d_0 \leq -18$$

since either $\varepsilon = 0$ or $d_0 \geq 6$. For $v \in V(G)$, define $\alpha(v) = 2$ if v has valency 6, and otherwise $\alpha(v) = 0$; and $\beta(v) = 1$ if v belongs to a 4-clique, and otherwise $\beta(v) = 0$. It follows that

$$\sum_{v\in V(G)} (\alpha(v) + \beta(v)) \leq 14$$

since there are ≤ 10 vertices in 4-cliques by (4.5), and ≤ 2 6-valent vertices by (5.6). For $1 \leq i \leq t$, let

$$f_i = \alpha(u) + \beta(u) + \alpha(v) + \beta(v)$$

where e_i has ends uv. Hence $\sum_{1\leq i\leq t} f_i \leq 14$, and so

$$\sum_{1\leq i\leq t} (d_i + f_i - 2T_i - 10) - \sum_{1\leq i,j\leq t} S_{ij} \leq -4.$$

For $1 \leq i \leq t$, let $S_i = \sum_{1\leq j\leq t} S_{ij}$, and let $R_i = d_i + f_i - 2T_i - 10$. Then

$$-4 \geq \sum_{1\leq i\leq t} R_i - \sum_{1\leq i,j\leq t} S_{ij} = \sum_{1\leq i\leq t: S_i=0} R_i + \sum_{1\leq i\leq t: S_i>0} R_i - \sum_{1\leq i,j\leq t} S_{ij}.$$

Suppose first that $\sum(R_i : 1 \leq i \leq t, S_i = 0) < 0$. Choose i with $S_i = 0$ and $R_i < 0$; $i = 1$ say. Let e_1 have ends ab. Since $R_1 < 0$,

$$d_1 + f_1 \leq 2T_1 + 9.$$

But $d_1 + f_1 \geq 14$, because $\delta(v) + \alpha(v) \geq 7$ for every vertex v, and so $2T_1 \geq 5$. Hence $T_1 \geq 3$, and so $T_1 = 3$ by (2.7). Consequently, $d_1 + f_1 \leq 15$. If a is 6-valent, then $\alpha(a) + \beta(a) \geq 3$, and so $d_1 + f_1 \geq 16$, a contradiction. Thus $\delta(a) \geq 7$, and similarly $\delta(b) \geq 7$. But $\delta(a) + \delta(b) + \beta(a) + \beta(b) \leq 15$, and so (i) or (ii) holds.

We may therefore assume that $\sum(R_i : 1 \leq i \leq t, S_i = 0) \geq 0$. Consequently,

$$\begin{aligned} -4 &\geq \sum_{1\leq i\leq t: S_i>0} R_i - \sum_{1\leq i,j\leq t} S_{ij} \\ &= \sum \left(\left(\frac{R_i}{S_i} - 1 \right) S_{ij} : 1 \leq i,j \leq t, S_{ij} = 1 \right) \\ &= \frac{1}{2} \sum \left(\left(\frac{R_i}{S_i} + \frac{R_j}{S_j} - 2 \right) S_{ij} : 1 \leq i,j \leq t, S_{ij} = 1 \right). \end{aligned}$$

We may therefore choose i,j with $S_{ij} = 1$ such that $\frac{R_i}{S_i} + \frac{R_j}{S_j} - 2 < 0$; and by exchanging i,j we may assume that $\frac{R_i}{S_i} - 1 < 0$. Let $i = 1$, $j = 2$ say, and let e_1 have ends ab and e_2 have ends cd. Since $S_{ij} = 1$, we may assume that a is adjacent to c and b to d, and ad, bc are not adjacent.

Now $\frac{R_1}{S_1} - 1 < 0$, and so $d_1 + f_1 - 2T_1 - S_1 \leq 9$. By (11.4), $T_1 \leq 2$, since $S_1 \geq 1$. Suppose that $T_1 \leq 1$. Since $d_1 + f_1 \geq 14$, we deduce that $S_1 \geq 3$ if $T_1 = 1$, and $S_1 \geq 5$ if $T_1 = 0$, and hence G has an F_7- or F_8-subgraph, contrary to (11.10). Thus $T_1 = 2$. By (11.10), G has no F_6-subgraph, and so $S_1 = 1$.

Thus $d_1 + f_1 \leq 14$. But $\delta(a) + \alpha(a) + \beta(a) \geq 7$, with equality only if $\delta(a) = 7$, and similarly for b. Hence a and b are both 7-valent, and consequently $\beta(a) = \beta(b) = 0$, and $e_1 \in M$.

If we replace e_1 and e_2 in the matching $e_1, \ldots, e_t$ by the edges ac and bd, we obtain another matching of the same cardinality; and therefore from the minimality of $|\{e_1, \ldots, e_t\} \cap M|$, we may assume that the edge ac belongs to M. Consequently, ac is in two triangles, and c is 7-valent, and $\beta(c) = 0$.

Now $\frac{R_1}{S_1} + \frac{R_2}{S_2} - 2 < 0$. We have shown that $S_1 = 1$ and $R_1 = d_1 + f_1 - 2T_1 - 10 = 0$. Consequently, $\frac{R_2}{S_2} < 2$, and so

$$7 + \delta(d) + \alpha(d) + \beta(d) - 2T_2 - 10 < 2S_2,$$

that is, $S_2 + T_2 \geq \frac{1}{2}(\delta(d) + \alpha(d) + \beta(d) - 2)$. Suppose that $\delta(d) + \alpha(d) + \beta(d) \geq 9$; then $S_2 + T_2 \geq 4$, contrary to (11.4) and (11.10). Thus $\delta(d) + \alpha(d) + \beta(d) \leq 8$. Hence d has valency 7 or 8, and if it is 8-valent then $\beta(d) = 0$. Thus (iii) holds. ∎

(13.2) *Let G be a non-apex Hadwiger graph; then* (13.1)(i) *does not hold.*

Proof. Suppose that $a, b \in V(G)$ are adjacent, and a has valency 7, and b has valency 8, and ab is in three triangles, and neither a nor b is in a 4-clique. Let a have neighbours $b, x_1, x_2, x_3, a_1, a_2, a_3$ and let b have neighbours $a, x_1, x_2, x_3, b_1, b_2, b_3, b_4$. Since a is not in a 4-clique, $\{x_1, x_2, x_3\}$ is a stable set, and some two of a_1, a_2, a_3 are not adjacent, say a_1a_2. For $1 \leq i \leq 3$, at most one of $b_1, \ldots, b_4$ is adjacent to x_i; for if b_1, b_2 say are both adjacent to x_i then $G \mid \{a, b, x_1, x_2, x_3, b_1, b_2\}$ has an F_5-subgraph, contrary to (11.4). We may therefore assume that b_1 is not adjacent to any of x_1, x_2, x_3, and so $\{b_1, x_1, x_2, x_3\}$ is stable. By (5.1) taking $X_1 = \{a_1, a, a_2\}$ and $X_2 = \{b, b_1, x_1, x_2, x_3\}$, there is a 5-colouring ϕ of $G \setminus \{a, b\}$ such that $\phi(a_1) = \phi(a_2)$ and $\phi(b_1) = \phi(x_1) = \phi(x_2) = \phi(x_3)$. Choose $\beta \in \{1, \ldots, 5\}$ with $\beta \neq \phi(b_1), \phi(b_2), \phi(b_3), \phi(b_4)$; and choose $\alpha \in \{1, \ldots, 5\}$ with $\alpha \neq \beta, \phi(a_1), \phi(a_3), \phi(x_1)$. Then setting $\phi(b) = \beta, \phi(a) = \alpha$ defines a 5-colouring of G, a contradiction. ∎

(13.3) *Let G be a non-apex Hadwiger graph; then* (13.1)(ii) *does not hold.*

Proof. Suppose that $a, b \in V(G)$ are adjacent, both of valency 7, and ab is in three triangles, and a is not in a 4-clique. Let a have neighbours $b, x_1, x_2, x_3, a_1, a_2, a_3$, and let b have neighbours $a, x_1, x_2, x_3, b_1, b_2, b_3$. Since a is not in a 4-clique, $\{x_1, x_2, x_3\}$

is stable, and some two of a_1,a_2,a_3 are not adjacent, say a_1,a_2. By (5.1) taking $X_1=\{a_1,a,a_2\}$ and $X_2=\{b,x_1,x_2,x_3\}$, there is a 5-colouring ϕ of $G\setminus\{a,b\}$ such that $\phi(a_1)=\phi(a_2)$ and $\phi(x_1)=\phi(x_2)=\phi(x_3)$. Choose $\beta\in\{1,\dots,5\}$ with $\beta\neq\phi(b_1),\phi(b_2),\phi(b_3)$, $\phi(x_1)$; and choose $\alpha\in\{1,\dots,5\}$ with $\alpha\neq\beta,\phi(a_1),\phi(a_3),\phi(x_1)$. Setting $\phi(a)=\alpha$ and $\phi(b)=\beta$ defines a 5-colouring of G, a contradiction. ∎

(13.4) *Let G be a non-apex Hadwiger graph, and let $a,b\in V(G)$ be distinct and both 7-valent, such that a is in no 4-clique. Then there are ≤ 3 vertices adjacent to both a and b.*

Proof. Suppose that $x_1,x_2,x_3,x_4\in V(G)-\{a,b\}$ are distinct and all adjacent to both a and b. By (11.4) no other vertex is adjacent to both a and b, and by (2.7) ab are not adjacent. Let a have neighbours $x_1,x_2,x_3,x_4,a_1,a_2,a_3$, and let b have neighbours $x_1,x_2,x_3,x_4,b_1,b_2,b_3$.

(1) *None of a_1,a_2,a_3,b_1,b_2,b_3 is adjacent to any of x_1,x_2,x_3,x_4.*

For if a_1x_1 are adjacent, say, then $G\mid\{a,b,a_1,x_1,x_2,x_3,x_4\}$ has an F_4-subgraph, contrary to (11.4).

Since a is in no 4-clique, no three of x_1,x_2,x_3,x_4 are mutually adjacent, and so we may express $\{x_1,x_2,x_3,x_4\}=Y_1\cup Y_2$ where $Y_1\cap Y_2=\emptyset$ and Y_1,Y_2 are stable. By (1) and (5.1), taking $X_1=Y_1\cup\{a,a_1\},X_2=Y_2\cup\{b,b_1\}$, there is a 5-colouring ϕ of $G\setminus\{a,b\}$ such that $\phi(y)=\phi(a_1)$ for all $y\in Y_1$, and $\phi(y)=\phi(b_1)$ for all $y\in Y_2$. Choose $\alpha\in\{1,\dots,5\}$ with $\alpha\neq\phi(a_1),\phi(a_2),\phi(a_3),\phi(b_1)$, and choose $\beta\in\{1,\dots,5\}$ with $\beta\neq\phi(b_1),\phi(b_2),\phi(b_3),\phi(a_1)$. Setting $\phi(a)=\alpha,\phi(b)=\beta$ defines a 5-colouring of G, a contradiction. ∎

We need the following

(13.5) *Let I_1,I_2,I_3,I_4 be four sets, each of cardinality ≥ 2. Then there exist $\alpha_i\in I_i\,(1\leq i\leq 4)$ such that $\alpha_1\neq\alpha_2\neq\alpha_3\neq\alpha_4\neq\alpha_1$.*

Proof. If $I_1=I_2=I_3=I_4$, let $a,b\in I_1$ be distinct and let $\alpha_1=\alpha_3=a$ and $\alpha_2=\alpha_4=b$. Thus we may assume that $I_1\not\subseteq I_4$. Choose $\alpha_1\in I_1-I_4$. Choose $\alpha_2\in I_2-\{\alpha_1\}$, $\alpha_3\in I_3-\{\alpha_2\}$, $\alpha_4\in I_4-\{\alpha_3\}$; then $\alpha_4\neq\alpha_1$, since $\alpha_1\notin I_4$. ∎

Finally, we complete the proof, by showing

(13.6) *Every Hadwiger graph is apex.*

Proof. Suppose G is a non-apex Hadwiger graph. By (13.1), (13.2) and (13.3), (13.1)(ii) holds; let $a,b,c,d\in V(G)$ be distinct, such that ab,bd,ac,cd are adjacent, ad,bc are not adjacent, ab,ac are both in two triangles, a,b,c are 7-valent and are in no 4-clique, and either d has valency 7, or d has valency 8 and is in no 4-clique.

(1) *There is a vertex p adjacent to a, b and c, and no vertex except a, d and p is adjacent to both b and c.*

For if there is no such vertex p, then since ab is in 2 triangles and so is ac, there are $u,v,w,x\in V(G)$ such that u,v,w,x,a,b,c,d are distinct and ua,ub,va,vb,wa,wc,xa,xc are edges, forming an F_9-subgraph contrary to (11.11). Thus, there is such a vertex p. The second claim follows from (13.4).

(2) *There are vertices q, r so that a,b,c,d,p,q,r are distinct, and q is adjacent to a and b, and r is adjacent to a and c.*

For ab is in two triangles, and so there exists a vertex $q \neq p$ adjacent to both a and b. Then $q \neq c,d$ since ad,bc are not adjacent. Similarly there exists $r \neq a,b,c,d,p$ adjacent to a and c. By (1), $q \neq r$.

(3) *q and r are not adjacent to p or d.*

For if dq are adjacent, then G has an F_5-subgraph with vertex set $\{a,b,c,d,p,q,r\}$ (delete the edges bp and bq). So dq and similarly dr are non-adjacent. Clearly pq are not adjacent, since a is in no 4-clique, and similarly pr are not adjacent.

From (1), (2) and (3), the only pairs among a,b,c,d,p,q,r whose adjacency is so far undecided are qr and dp.

(4) *qr are not adjacent.*

For suppose that they are. Since b has valency 7 and is not in a 4-clique, there are neighbours x,y of b with $x,y \neq a,d,p,q$ such that xy are not adjacent. Then $x,y \neq c,r$ since c,r are not adjacent to b. By (5.2), $\{x,y,a,d\}$ is not stable. But ax are not adjacent since the edge ab is in ≤ 2 triangles by (13.3), and similarly ay are not adjacent, and so we may assume that dx are adjacent. But then $G \mid \{a,b,c,d,p,q,r,x\}$ has an F_{10}-subgraph (delete ap,aq,ar). By (11.12), some $v \in \{a,p,x\}$ has no neighbour in $V(G) - \{a,b,c,d,p,q,r,x\}$. Now $v \neq a$ since a is 7-valent and ad are not adjacent; $v \neq p$ since pq and pr are not adjacent, by (3); and $v \neq x$ since xa are not adjacent as we already saw, and xc are not adjacent by (1). This is a contradiction, and (4) follows.

(5) *dp are adjacent.*

For suppose they are not. Then $\{d,p,q,r\}$ is stable. Let a have neighbours b,c,p,q, r,a_1,a_2; then $a_1,a_2 \neq d$. Let b have neighbours a,d,p,q,b_1,b_2,b_3; then $b_1,b_2,b_3 \neq r,c$. Let c have neighbours a,d,p,r,c_1,c_2,c_3; then $c_1,c_2,c_3 \neq b,q$. Since b is in no 4-clique we may assume that b_1b_2 are non-adjacent. By (5.1) with $X_1 = \{b,b_1,b_2\}$ and $X_2 = \{a,c,p,q,r,d\}$, there is a 5-colouring ϕ of $G \setminus \{a,b,c\}$ such that $\phi(b_1) = \phi(b_2)$ and $\phi(p) = \phi(q) = \phi(r) = \phi(d)$. Choose $\alpha_1 \in \{1,\ldots,5\}$ with $\alpha_1 \neq \phi(c_1),\phi(c_2),\phi(c_3)$, $\phi(p)$; choose $\alpha_2 \in \{1,\ldots,5\}$ with $\alpha_2 \neq \alpha_1,\phi(a_1)$, $\phi(a_2),\phi(p)$; and choose $\alpha_3 \in \{1,\ldots,5\}$ with $\alpha_3 \neq \alpha_2,\phi(b_1),\phi(b_3)$, $\phi(p)$. Setting $\phi(c) = \alpha_1, \phi(a) = \alpha_2$, $\phi(b) = \alpha_3$ defines a 5-colouring of G, a contradiction. This proves (5).

(6) *There is a vertex $s \notin \{a,b,c,d,p,q,r\}$ adjacent to b and d; and a vertex $t \notin \{a,b,c,d,p,q,r\}$ adjacent to c and d. Moreover, $s \neq t$.*

For let b_1,b_2 be two non-adjacent neighbours of b with $b_1,b_2 \neq a,b,c,d,p,q,r$. By (5.2), $\{a,b_1,b_2,d\}$ is not stable, and yet ab_1 and ab_2 are not adjacent, because the edge ab is in ≤ 2 triangles, by (13.3). Thus one of b_1,b_2 is adjacent to d, and so there is such a vertex s, and similarly t; and $s \neq t$ by (1). This proves (6).

(7) *s is not adjacent to any of a, c, p, q, r, t; and t is not adjacent to any of a, b, p, q, r, s.*

For sa are not adjacent since by (13.3) the edge ab is in ≤ 2 triangles; sc are not adjacent by (1); sp are not adjacent since b is in no 4-clique; and sr are not adjacent for otherwise G would have an F_6-subgraph with vertex set $\{a,b,c,d,p,q,r,s\}$ (delete cp,dp,rc,sd). It remains to check sq and st. Suppose that sq are adjacent; then G has an F_{10}-subgraph with vertex set $\{a,b,c,d,p,q,r,s\}$ (delete bp,bq,bs,cp) and so by (11.12), some $v \in \{b,p,r\}$ has no neighbour in $V(G)-\{a,b,c,d,p,q,r,s\}$. But $v \neq b$ since b is 7-valent and bc are not adjacent; $v \neq p$ since pq and pr are not adjacent; and $v \neq r$ since pr and qr are not adjacent. This shows that sq are not adjacent. Similarly, t is not adjacent to any of a,b,p,q,r.

Now suppose that st are adjacent. Then G has an F_{10}-subgraph with vertex set $\{a,b,c,d,p,q,s,t\}$ (delete ap,dp,ds,dt). By (11.12) some $v \in \{p,q,d\}$ has no neighbour in $V(G) - \{a,b,c,d,p,q,s,t\}$. Now $v \neq p$ since pq,ps are not adjacent; $v \neq q$ since qc,qd are not adjacent; and $v \neq d$ since da,dq are not adjacent, a contradiction. Thus st are not adjacent. This proves (7).

Let a_1,a_2 be the two neighbours of a not in $\{a,b,c,d,p,q,r,s,t\}$ and define b_1,b_2 for b and c_1,c_2 for c similarly. Now d may have valency 7 or 8. Let N be the set of two or three neighbours of d not in $\{a,b,c,d,p,q,r,s,t\}$. If $|N| = 3$ then d is 8-valent and so not in a 4-clique; and therefore, whether $|N| = 2$ or 3, there is a stable subset $Y \subseteq N$ with $|N - Y| = 1$. Let $N - Y = \{d_1\}$ and let $d_2 \in Y$. By (5.1) with $X_1 = \{a,b,c,p,q,r,s,t\}$ and $X_2 = Y \cup \{d\}$, there is a 5-colouring ϕ of $G \setminus \{a,b,c,d\}$ such that

$$\phi(p) = \phi(q) = \phi(r) = \phi(s) = \phi(t)$$

and $\phi(y) = \phi(d_2)$ for all $y \in Y$. Let

$$\begin{aligned}
I(a) &= \{1,\dots,5\} - \{\phi(a_1),\phi(a_2),\phi(p)\}\\
I(b) &= \{1,\dots,5\} - \{\phi(b_1),\phi(b_2),\phi(p)\}\\
I(c) &= \{1,\dots,5\} - \{\phi(c_1),\phi(c_2),\phi(p)\}\\
I(d) &= \{1,\dots,5\} - \{\phi(d_1),\phi(d_2),\phi(p)\}.
\end{aligned}$$

By (13.5) there exist $\alpha_1 \in I(a), \alpha_2 \in I(b)$, $\alpha_3 \in I(d)$, $\alpha_4 \in I(c)$ such that $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_4 \neq \alpha_1$. Then setting $\phi(a) = \alpha_1, \phi(b) = \alpha_2$, $\phi(d) = \alpha_3, \phi(c) = \alpha_4$ defines a 5-colouring of G, a contradiction. This completes the proof. ∎

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